1. Let $\mathbb{N}$ be the set of positive integers. Let $\mathcal{T}$ be the collection of subsets $U \subset \mathbb{N}$ satisfying the following: if $x \in U$ and x is odd, then also $x+1 \in U$.

   (a) [2 pts] Prove that $\mathcal{T}$ is a topology on $\mathbb{N}$.

   For the rest of this problem, $\mathbb{N}$ denotes the set of positive integers with the topology $\mathcal{T}$.

   (b) [1 pt] Describe the closed subsets of $\mathbb{N}$.

   Answer: A subset $C \subset \mathbb{N}$ is closed if for every y such that y is even and $y \in C$, $y-1 \in C$. Proof: C is closed if and only if $\mathbb{N} \setminus C$ is open. The set $\mathbb{N} \setminus C$ is open if and only if for every odd x, if $x \notin C$ then $x+1 \notin C$. Stating in in the contrapositive form, this is equivalent to saying that for every odd x such that $x+1 \in C$, $x \in C$. This is equivalent to saying that for every even $y \in C$, $y-1 \in C$.

   (c) [2 pts] Describe the connected components of $\mathbb{N}$.

   Answer: The connected components are sets of the form $\{x, x+1\}$, where x is odd. Proof: First, notice that every set of this form is both open and closed. Second, notice that this set is connected. The only possible separation is $\{x\}, \{x+1\}$, and it is not a separation because $\{x\}$ is not open. Third, notice that $\mathbb{N}$ is the disjoint union of sets of this form, so sets of these forms define a partition of $\mathbb{N}$ into sets that are open and closed. It follows that every connected component of $\mathbb{N}$ is a subset of $\{x, x+1\}$ for some odd x. But these sets are themselves connected, so they are the connected components.

   (d) [1 pt] Describe the path components of $\mathbb{N}$.

   Answer: The path components are the same as the connected components. For this it is enough to prove that the connected components are path-connected. Let x be odd. The map $\alpha\colon [0,1] \to \{x, x+1\}$ defined by $f(0) = x, f(t) = x+1$ for all $0 < t \leq 1$, is a path from x to $x+1$.

   (e) [1 pt] Is $\mathbb{N}$ locally connected?

   Answer: Yes. Let $x \in \mathbb{N}$ and $x \in U \subset \mathbb{N}$ be an open neighborhood. We need to prove that there is a connected open neighborhood $x \in V \subset U$. If x is even then $\{x\}$ is open, and we can take $V = \{x\}$. If x is odd, then $\{x, x+1\} \subset U$ because U is open. We have seen that $\{x, x+1\}$ is open and connected, so we can take $V = \{x, x+1\}$.

2. Define an equivalence relation on $\mathbb{R}^n \times \{1,2\}$ by declaring that $(\bar{x}, 1) \sim (\bar{x}, 2)$ for all **nonzero** vectors $\bar{x} \in \mathbb{R}^n \setminus \{\bar{0}\}$. Let X be the quotient of $\mathbb{R}^n \times \{1,2\}$ by this relation, and let $q\colon \mathbb{R}^n \times \{1,2\} \to X$ be the quotient map.

   (a) [2 pts] Is q an open map?

   Answer: Yes. Proof: Let $U \subset R^n \times \{1,2\}$ be an open subset. Then U decomposes as a disjoint union $U = U_1 \coprod U_2$, where U_i is an open subset of $\mathbb{R}^n \times \{i\}$. We want to prove that $q(U)$ is open. For this, we need to prove that $q^{-1}(q(U))$ is an open subset of $\mathbb{R}^n \times \{1,2\}$. By definition of the equivalence relation on $\mathbb{R}^n \times \{1,2\}$, we have

   $$q^{-1}(q(U)) = U_1 \cup U_1^* \cup U_2 \cup U_2^*,$$

   where U_1^* is the copy in $\mathbb{R}^n \times \{2\}$ of $U_1 \setminus \{\bar{0}\}$. Similarly, U_2^* is the copy in $\mathbb{R}^n \times \{1\}$ of $U_2 \setminus \{\bar{0}\}$. Since $\{\bar{0}\}$ is closed in $\mathbb{R}^n$, it follows that U_1^* and U_2^* are open. It follows that $q^{-1}(q(U))$ is the union of four open sets, so it is open.

(b) [1 pt] Is q a closed map?

Answer: No. Proof: Let $D^n \subset \mathbb{R}^n$ be the closed unit ball. Then $D^n \times \{1\}$ is a closed subset of $\mathbb{R}^n \times \{1, 2\}$. On the other hand, $q^{-1}(q(D^n \times \{1\})) = D^n \times \{1\} \cup (D^n \setminus \{\bar{0}\}) \times \{2\}$. This is not a closed subset of $\mathbb{R}^n \times \{1, 2\}$, therefore $q(D^n \times \{1\})$ is not a closed subset of X.

(c) [2 pts] Is X locally Euclidean?

Answer: Yes. To prove this, we will prove that q is a local homeomorphism. We saw already that q is an open map. It also is locally one to one. By this we mean that every point z of $\mathbb{R}^n \times \{1, 2\}$ has an open neighborhood U, such that q is one to one on U. Indeed, one can take U to be either $\mathbb{R}^n \times \{1\}$ or $\mathbb{R}^n \times \{2\}$, depending on which one contains z. Then q is one to one on U, and for every open subset $V \subset U$, $q(V)$ is an open subset of X. It follows that q induces a homeomorphism between U and $q(U)$. But every point of X has a neighborhood of the form $q(U)$, where U is an open subset of $\mathbb{R}^n \times \{1\}$ or of $\mathbb{R}^n \times \{2\}$. It follows that X is locally Euclidean.

(d) [1 pt] Is X Hausdorff?

Answer: No. The points $(\bar{0}, 1)$ and $(\bar{0}, 2)$ can not be separated by open sets.

3. Let $f: X \to Y$ be a continuous map. Recall that a retraction of f is a continuous map $r: Y \to X$ such that $r \circ f$ is the identity on X. Similarly, a section of f is a continuous map $s: Y \to X$ such $f \circ s$ is the identity on Y.

   Determine whether

   (a) [1 pt] The inclusion $(1, 2) \hookrightarrow [0, 3]$ has a retraction.

   Answer: No. A retraction would be a surjective map $r: [0, 3] \to (1, 2)$. But $[0, 3]$ is compact while $(1, 2)$ is not, so no such surjective map exists.

   (b) [1 pt] The inclusion $[1, 2] \to (0, 3)$ has a retraction.

   Answer: Yes. The map $r: (0, 3) \to [1, 2]$ defined by $r(x) = 1$ for $x \leq 1$, $r(x) = x$ for $1 \leq x \leq 2$ and $r(x) = 2$ for $x \geq 2$ is a retraction.

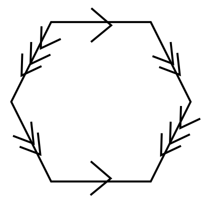
   (c) [1 pt] The equatorial inclusion $S^1 \hookrightarrow S^2$ has a retraction.

   Answer: No, because a retraction would induce an injective homomorphism $\mathbb{Z} \cong \pi_1(S^1) \to \pi_2(S^2) = 0$, which is impossible.

   (d) [2 pts] The squaring map $f(z) = z^2$ considered as a map from $\mathbb{C}$ to itself has a section.

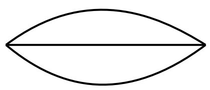
   Answer: No. If f has a section then the restriction of f to the unit circle (which is also the preimage of the unit circle) also would have a section. But the restriction of f to the unit circle is a two-fold covering map. A two-fold covering map is not surjective on π_1 (it induces an inclusion onto a subgroup of index 2), so it does not have a section.

4. Let X be the quotient space of the hexagon obtained by identifying opposite pairs of edges, as indicated in the following picture



(a) [1 pt] The standard CW structure on the hexagon, with six 0-cells, six 1-cells and a single 2-cell induces a CW-structure on X. How many cells in each dimension does X have with this structure? Draw a picture of the 1-skeleton of X.

Answer: 2 zero-dimensional cells, 3 one-dimensional cells, 1 two-dimensional cell. Here is a picture of the 1-skeleton



(b) [1 pt] What is the Euler characteristic of X?

Answer: $2 - 3 + 1 = 0$

(c) [1 pt] Is X a surface? If yes, identify it with a familiar surface, if not, explain why it is not.

Answer: Yes, X is a surface because every edge is identified with exactly one other edge. Since there is no reversal of direction, the surface is orientable. Since the Euler characteristic is zero, it has to be the Torus.

(d) [1 pt] Describe the fundamental group of X.

Answer: The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

5. (a) [2 pts] Prove that the space $\mathbb{R}^4 \setminus \{\bar{0}\}$ is simply-connected.

Answer: The space is easily shown to be homotopy equivalent to S^3, which is simply-connected.

Now consider the homeomorphism $f\colon \mathbb{R}^4\setminus\{\bar{0}\}) \to \mathbb{R}^4\setminus\{\bar{0}\}$ defined by the formula $f(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$. Notice that $f \circ f \circ f \circ f$ is the identity map.

(b) [2 pts] Prove that f induces a covering space action of $\mathbb{Z}/4$ on $R^4 \setminus \{\bar{0}\}$. In other words, prove that there is a covering space action of $\mathbb{Z}/4$ on $R^4 \setminus \{\bar{0}\}$, where the generator of $\mathbb{Z}/4$ acts by the map f.

Answer: f is clearly a homeomorphism. Since $f^{\circ 4}$ is the identity, f generates a cyclic subgroup of order 4 in the group of homeomorphisms of $\mathbb{R}^4 \setminus \{0\}$. This is equivalent to saying that f induces an action of $\mathbb{Z}/4$ on this space. It is easy to check that f, f^2, f^3 have no fixed points. It follows that the action of $\mathbb{Z}/4$ on $\mathbb{R}^4 \setminus \{0\}$ induced by f is free. A free action of a finite group on a Hausdorff space is a covering space action.

Let $X = (R^4 \setminus \{\bar{0}\})/_{\mathbb{Z}/4}$ be the quotient space of this action. Remember that even if you did not do parts (a) and (b) of the problem, you still can assume them in the remaining parts.

(c) [1 pt] Find the fundamental group of X.

Answer: X is the quotient space of a covering space action of $\mathbb{Z}/4$ on a simply-connected space. It follows that $\pi_1(X) \cong \mathbb{Z}/4$.

(d) [1 pt] How many pairwise non-isomorphic covering spaces does X have?

Answer: Covering spaces over X are indexed by conjugacy classes of subgroups of $\mathbb{Z}/4$. There are three such conjugacy classes: the trivial group, $\mathbb{Z}/2$, and all of $\mathbb{Z}/4$.

(e) [2 pts] Describe the automorphism group of each of the covering spaces of X that you found in the previous part.

Answer: The automorphism group of a covering space corresponding to a subgroup $H \subset \mathbb{Z}/4$ is the quotient of the normalizer of H by H. The group that we get are $\mathbb{Z}/4$, $\mathbb{Z}/4/\mathbb{Z}/2 \cong \mathbb{Z}/2$, and the trivial group.