MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Gregory Arone

Tentamensskrivning i
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(1) Let S_1 and $S_2 \subseteq \mathbb{R}^2$ two circles of radii 1 and 2 respectively. Let $A := S_1 \cup S_2$ and consider the following family of subsets of A:

$$\mathcal{T} := \{U \mid U \text{ is open with the induced euclidean topology AND } 2x \in U \text{ whenever } x \in U \cap S_1\}$$

(a) Show that $\mathcal{T}$ is a topology on A.
(b) Determine interior, closure, exterior, and boundary of S_1 in the topology $\mathcal{T}$.
(c) Say for which $a \in A$ the singlet $\{a\}$ is closed in the topology $\mathcal{T}$.
(d) Determine if the topological space $(A, \mathcal{T})$ is connected.
(e) Determine whether the topological space $(A, \mathcal{T})$ is compact.
(f) Determine whether the topological space $(A, \mathcal{T})$ is Hausdorff (T_2). [30 points]

Solution:
(a) Both A and $\emptyset$ are open in the induced Euclidean topology (since it is a topology). For the empty set, the second requirement is void, for A it is trivially satisfied, so we have that A and $\emptyset$ are open. Let now $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ an arbitrary family of element of $\mathcal{T}$ since they are open in the induced Euclidean topology we have that $\bigcup U_\alpha$ is open in the Euclidean topology. Suppose now that $x \in S_1$ belongs to $\bigcup U_\alpha$ then there is an $\alpha \in \mathcal{A}$ such that $x \in U_\alpha$. By the definition of $\mathcal{T}$ we have that $2x \in U_\alpha \subseteq \bigcup U_\alpha$. So $\bigcup U_\alpha$ is open in $\mathcal{T}$. It remains to show that finite intersection of open sets are open. We can limit ourselves to consider the case of the intersection of two elements, the general situation will follow by induction. Thus let U and V in $\mathcal{T}$, as they are open in the induced Euclidean topology we have that $U \cap V$ is open in the induced Euclidean topology. Hence we just need to check that it satisfies the second requirement for $\mathcal{T}$. Supoose that $x \in S_1$ is in $U \cap V$, then x belongs both to U and V, and, by the definition of $\mathcal{T}$ we also have that $2x \in U \cap V$. Therefore $U \cap V$ is open.
(b) Let U a non empty open set. If $U \cap S_1 \neq \emptyset$ then clearly $U \cap S_2 \neq \emptyset$. In particular non empty open sets cannot be entirely contained in S_1. We deduce that the interior of S_1 is empty. On the other side we have that the closure of S_1 is the whole S_1. In fact we can show that $S_2 = A \backslash S_1$ is open in $\mathcal{T}$. For any $y \in S_2$ consider the ball $B_{\frac{1}{2}}(y) \subseteq \mathbb{R}^2$. We have that $B_{\frac{1}{2}}(y) \cap S_1 = \emptyset$ and so we can write $S_2 = \bigcup_{y \in S_2} B_{\frac{1}{2}}(y) \cap A$. The sets $B_{\frac{1}{2}}(y) \cap A$ are clearly open in A with respect to the induced Euclidean topology. In addition, as they do not intersect S_1 they trivially satisfy the second condition for $\mathcal{T}$. So they are open in $\mathcal{T}$ and S_2 is open as it is a union of open sets. The exterior of S_1 is

$$\text{Ext}(S_1) = A \backslash \overline{S_1} = S_2.$$

The boundary of S_1 is

$$\partial S_1 = \overline{S_1} \backslash \text{Int}(S_1) = S_1.$$

(c) If $a \in S_1$, then we have that $\{a\}$ is closed. In fact $U := A \backslash \{a\}$ is open in the Euclidean topology, and if $x \in S_1 \cap U$, $2x \neq a$, so $2x \in U$. Conversely if a does not belong to S_1 then $\{a\}$ cannot be closed. Otherwise we would

have that $A\backslash\{a\}$ is open. Since $a \in S_2$ we can write $a = 2b$ with b in S_1. We then will have that $b \in A\backslash\{a\}$ but $2b = a \notin A\backslash\{a\}$. Thus $A\backslash\{a\}$ does not satisfy the second requirement for membership to $\mathcal{T}$.
(d) The space A is connected in the given topology. In order to see this we begin observing that, as $\mathcal{T}$ is coarser than the Euclidean topology we have that the subspace topology induced by $\mathcal{T}$ on S_1 and S_2 is coarser than the induced Euclidean topology. Both S_1 and S_2 are connected with respect to the Euclidean topology as they are homoemorphic to $\mathbb{S}^1$, since connectedness is stable with respect to taking coarser topology we deduce that S_1 and S_2 are connected with respect of $\mathcal{T}$. Suppose now that $f : A \to \{0,1\}$ is a continuous map where $\{0,1\}$ is endowed with the discrete topology. We want to show that this is constant. Since the spaces S_i's are connected we have that, for every $x \in S_1$, $f(x) = f(0,1)$ and for every $y \in S_2$, $f(y) = f(0,2)$. We will show that $f(0,1) = f(0,2)$, which will imply that f is constant. By the continuity of f we have that $f^{-1}(\{f(0,1)\})$ is an open set containing $(0,1)$. The definition of $\mathcal{T}$ yields that $(0,2) \in f^{-1}(\{f(0,1)\})$ and so $f(0,2) = f(0,1)$.
(e) The topology $\mathcal{T}$ is coarser than the induced Euclidean topology and compactness is stable with respect to taking coarser topologies. The set A is closed and bounded with respect to the Euclidean topology, so it is a compact topological space with the induced euclidean topology. We deduce that A is compact with respect to $\mathcal{T}$.
(f) The space A with this topology is not Hausdorff. In fact let $x \in S_1$ and set $y = 2x$. Given any two open sets U_x and U_y with $x \in U_x$ and $y \in U_y$ for the definition of the topology $\mathcal{T}$ we have that $y \in U_x$. So $U_x \cap U_y \neq \emptyset$. One might also have observed that from (c) not all singlets in A are closed, but in an Hausdorff topological space all singlets are closed.

(2) Let X any topological space and let $A \subseteq X$ a subset. Consider the topological space $Y := X/A$ obtained from X by collapsing A, and let $\pi : X \to Y$ the quotient map.
(a) Show that if $C \subseteq X$ is closed and $C \cap A = \emptyset$, then $\pi(C)$ is closed.
[20 points] (b) Show that if Y is Hausdorff (T_2) then A is closed.

Solution:
(a) Let Z a subset of Y. By the definition of quotient topology we have the following chain of equivalences

$$\begin{aligned} Z \text{ is closed} &\iff Y\backslash Z \text{ is open} \\ &\iff \pi^{-1}(Y\backslash Z) \text{ is open} \\ &\iff X\backslash \pi^{-1}(Z) \text{ is open} \\ &\iff \pi^{-1}(Z) \text{ is closed.} \end{aligned}$$

Thus, to show that $\pi(C)$ is closed we need to show that $\pi^{-1}(\pi(C))$ is closed. But if $A \cap C = \emptyset$ we have that $C = \pi^{-1}(\pi(C))$. In fact we always have the inclusion $C \subseteq \pi^{-1}(\pi(C))$. Suppose that $p \in \pi^{-1}(\pi(C))$. Then we can find $q \in C$ such that $\pi(q) = \pi(p)$. But then q is in relation with p. As $q \in C$ we have that $q \notin A$ and therefore, by the definition of the equivalence relation used to construct Y, we have that $q = p$.
(b) We prove the contrapositive statement. Suppose that A is not closed and let $p \in \overline{A}\backslash A$, and denote by p_A in Y the equivalence class of points in A, so that we have $\pi(a) = p_A$ for every $a \in A$. Given U and V open

neighbors of $\pi(p)$ and p_A respectively, we have, by the definition of quotient topology, that $\pi^{-1}(U)$ is an open neighbor of p in X. By the choice of p we have that $\pi^{-1}(U) \cap A \neq \emptyset$. Therefore we can find $a \in \pi^{-1}(U) \cap A$. Then we have that $p_A = \pi(a) \in U$ and $U \cap V$ is not empty. Therefore Y is not Hausdorff.

(3) Let X be the space $(\mathbb{S}^2 \times \mathbb{S}^2)\backslash\Delta_{\mathbb{S}^2}$, where $\Delta_{\mathbb{S}^2}$ is the diagonal.
  (a) Show that X is homotopy equivalent to $\mathbb{S}^2$.
  (b) Let $\mathbb{Z}/2\mathbb{Z}$ act on X by exchanging the coordinates. Compute the fundamental group of the quotient space $X/(\mathbb{Z}/2\mathbb{Z})$. [20 points]

Solution:
(a) Consider $Y := \{(x, -x) \,|x \in \mathbb{S}^2\} \subseteq X \subseteq \mathbb{S}^2 \times \mathbb{S}^2$. Let $p_1 : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$ the projection on the first factor. The restriction of p_1 to Y is clearly bijective, and since it is the restriction of a surjective open map this is also open, so it is an homeomorphism. We will show that Y is a strong deformation retract of X. Let $i : Y \hookrightarrow X$ the natural inclusion and $r : X \to Y$ to be the restriction of p_1 to X. Clearly we have that $r \circ i = \mathrm{id}_Y$. The map $H : X \times I \to X$ defined by

$$H((x, y), t) = \left(x, \frac{-tx + (1-t)y}{\|-tx + (1-t)y\|}\right)$$

yields an homotopy between id_X and $i \circ r$. In fact, as $y \neq x$ the segment joining y and $-x$ does not pass through the origin, and the map is continuous. We have to check that the target is effectively X. If we cut the sphere with the plane through x, $-x$ and y that are non co-linear points we see that H moves y toward $-x$ toward the smallest angle between the lines identified by y and x. This angle is strictly smaller than π so we see that $\frac{-tx+(1-t)y}{\|-tx+(1-t)y\|}$ cannot be equal to x.
(b) Observe that X is path connected, and hence also the quotient $X/(\mathbb{Z}/2\mathbb{Z})$ is path connected. Thus the fundamental groups do not depends from the base points. We know from the lectures that $\mathbb{S}^2$ is simply connected. By the homotopy invariance of the fundamental group we have that X is also simply connected. The action of $\mathbb{Z}/2\mathbb{Z}$ on X is a covering space action: let $(x, y) \in X$ and consider $U := (B_\rho(x) \times B_\rho(y)) \cap X$ with $\rho < \frac{d(x,y)}{3}$ where d is the euclidean distance in $\mathbb{R}^3$. Let $1+U$ be the image of $\sigma(1, -)$ over U, that is $1+U = \{(y, x) \mid (x, y) \in U\}$. We have that $U \cap (1+U) = \emptyset$. By the unicity of the universal cover we have that the quotient map $\pi : X \to X/(\mathbb{Z}/2\mathbb{Z})$ is the universal cover of $X/(\mathbb{Z}/2\mathbb{Z})$. We deduce that $\pi_1(X/(\mathbb{Z}/2\mathbb{Z})) \simeq \mathbb{Z}/2\mathbb{Z}$.

(4) Let X be a topological spaces and suppose that X can be written as the union of two simply connected open sets intersecting in a path connected space. Show that X is simply connected. [10 points]

Solution:
This is a direct application of SVK: we have that all the assumptions of the statement are verified as X can be written as $U \cup V$ with U and V open and $U \cap V$ path connected. In addition, as U and V are simply connected, they are in particular path connected, so X is path connected and the fundamental group of X (and those of U, V, and $U \cap V$) does not depend from the base point. SVK theorem yields that

$$\pi_1(X) \simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V),$$

which, independently of what $\pi_1(U \cap V)$ is, is the quotient of $\pi_1(U) * \pi_1(V)$ which is clearly trivial as U and V are simply connected. So $\pi_1(X)$ is trivial.

(5) (a) Define the universal covering space of a topological space.
(b) Give a sketch of the construction of the universal covering for a connected and locally simply connected space.

[20 points]