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Solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Programme, January 12, 2009

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1. Since $f'(x) = (x^2 - 2x - 3)e^x$ then $f'(x) = 0$ if and only if $x^2 - 2x - 3 = 0$, i.e. $x = 3$ or $x = -1$. As x lies between 0 and 4, the only solution is $x = 3$.
Now we have $f''(x) = (x^2 - 5)e^x$, thus $f''(3) > 0$, giving us $x = 3$ as a local minimum, $f(3) = -2e^3$. At the ends of the interval we find that $f(0) = 1$ and $f(4) = e^4$. The smallest value of the function is $-2e^3$ while the largest is e^4.

2. The derivative to $y = x^2$ at the point a is $2a$. The equation of the tangent line to $y = x^2$ at the point (a, a^2) is then $y = 2ax + m$ but since (a, a^2) lies on this line, we get $m = -a^2$, i.e. $y = 2ax - a^2$.
Similarly we find that the tangent line to $y = x^2 + 4x + 1$ at the point $(b, b^2 + 4b + 1)$ is $y = (2b + 4)x - b^2 + 1$. The both tangents overlap if and only if $2a = 2b + 4$ and $-a^2 = -b^2 + 1$. Solution to this sysyem of equations is $a = \frac{3}{4}$ and $b = -\frac{5}{4}$. The line L, the common tangent to both curves, has thus the equation $y = \frac{3}{2}x - \frac{9}{16}$

3. For the series $1 + \dfrac{1}{4}\ln^2 x + \dfrac{1}{16}\ln^4 x + \dfrac{1}{64}\ln^8 x + ...$ we have that $r = \dfrac{1}{4}\ln^2 x$ and since $|r|$ must be smaller than 1 then we have $|\dfrac{1}{4}\ln^2 x| < 1$. Since a square is never negative then $0 \leq \ln^2 x < 4$, implying that $-2 < \ln x < 2$. As the logarithmic function is monotone we get $e^{-2} < x < e^2$.

For the series $1 - 2e^{-x} + 4e^{-2x} - 8e^{-3x} + 16e^{-4x} - ...$ we have $r = -2e^{-x}$. Inequality $|-2e^{-x}| < 1$ reduces to $0 < e^{-x} < \dfrac{1}{2}$, since the exponential function is always positive. This impies that $-x < \ln\dfrac{1}{2} = -\ln 2$. Hence $x > \ln 2$.

Combining the two results we find that both series converge for $\ln 2 < x < e^2$.

4. In order to find the stationary points we solve the system of equations $f'_x = 0$ and $f'_y = 0$, i.e. $6x^2 + 12xy = 0$ and $6x^2 + 3y^2 - 12 = 0$. The first equation, $6x(x + 2y) = 0$ gives $x = 0$ or $x = -2y$. Substitution into the second equation gives in the first case $y = \pm 2$ and in the second case, after solving the quadratic equation $6(-2y)^2 + 3y^2 - 12 = 0$, $y = \pm\dfrac{2}{3}$ and $x = \mp\dfrac{4}{3}$. The four stationary point are: $P_1 = (0, 2)$, $P_2 = (0, -2)$, $P_3 = (\dfrac{4}{3}, -\dfrac{2}{3})$ and $P_4 = (-\dfrac{4}{3}, \dfrac{2}{3})$.
For each of those four points we study now $A = \dfrac{\partial^2 f}{\partial x^2} = 12(x + y)$, $B = \dfrac{\partial^2 f}{\partial y^2} = 6y$, $C = \dfrac{\partial^2 f}{\partial x \partial y} = 12x$ and $\Delta = AB - C^2$. We find that:
For P_1: $\Delta = 288 > 0$ and $A = 24 > 0$; thus P_1 is a local minimum.
For P_2: $\Delta = 288 > 0$ and $A = -24 < 0$; thus P_2 is a local maxmum.
For P_3 and P_4: $\Delta = -288 < 0$ and thus both points are saddle points..

5. a) $\displaystyle\int_1^e \frac{\ln x}{x(1 + \ln^2 x)}\,dx = ...$, by substitution $t = \ln x$; $dt = \dfrac{1}{x}dx$, $\displaystyle ... = \int_0^1 \frac{t}{1 + t^2}\,dt = \frac{1}{2}\int_0^1 \frac{2t}{1 + t^2}\,dt = \frac{1}{2}\ln(1 + t^2)\Big|_0^1 = \frac{1}{2}\ln 2.$

b) $\displaystyle\int_0^1 y^2 \cdot e^{2y}\,dy = ...$, integration by parts, $\displaystyle ... = y^2 \cdot \frac{1}{2}e^{2y}\Big|_0^1 - \int_0^1 2y \cdot \frac{1}{2}e^{2y}\,dy = \frac{1}{2}e^2 - \int_0^1 y \cdot e^{2y}\,dy = \frac{1}{2}e^2 - \left(y \cdot \frac{1}{2}e^{2y}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2y}\,dy\right) = \frac{1}{2}e^2 - \left(\frac{1}{2}e^2 - \frac{1}{4}e^{2y}\Big|_0^1\right) = \frac{1}{4}(e^2 - 1).$

6. The implicit differentiation gives $6xy + 3x^2y' + e^{x+y}(1 + y') + \dfrac{1}{x+y} \cdot (1 + y') = 0$. Putting in $x = 0$ and $y = y(0) = 1$, we get $e(1 + y'(0)) + (1 + y'(0)) = 0$, which is the same as $(e + 1)(1 + y'(0)) = 0$. Hence $y'(0) = -1$.
We find now the second derivative of the expression, which is: $6y + 6xy' + 6xy' + 3x^2y'' + e^{x+y}(1 + y')^2 + e^{x+y}y'' + \dfrac{y''(x + y) - (1 + y')^2}{(x + y)^2} = 0$. Letting now in $x = 0$, $y = y(0) = 1$ and $y' = y'(0) = -1$ we get $6 + ey''(0) + y''(0) = 0$, which implies that $y''(0) = -\dfrac{6}{e+1}$.

Paul