Exam in Mathematics for Economic and Statistical Analysis
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Solutions to the examination in Mathematics for Economic and Statistical Analysis

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1. Since $f'(x) = 4\dfrac{(x-1)}{(x+1)^3}$ and $g'(x) = (x+1)^3(5x-3)$ the equation reduces to $4(5x-3)(x-1) = 3(1-x)$. One solution is obviously $x = 1$. If $x \neq 1$ the we divide by $(1 - x)$ and we have $4(5x-3) = -3$. This gives the second solution, $x = \dfrac{9}{20}$.

2. Using the l'Hospiltals rule several times we get:

$$\lim_{x\to 0}\frac{x\ln(x+1)-x^2}{2(e^x-1-x)-x^2} = \lim_{x\to 0}\frac{\ln(x+1)+\dfrac{x}{x+1}-2x}{2(e^x-1)-2x} =$$

$$= \lim_{x\to 0}\frac{\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}-2}{2e^x-2} = \lim_{x\to 0}\frac{\dfrac{-1}{(x+1)^2}+\dfrac{-2}{(x+1)^3}}{2e^x} = -\frac{3}{2}.$$

In the second problem it is easiest to divide the numerator and the denominator by x^8.

$$\lim_{x\to\infty}\frac{10000x^7+1000x^5+100x^3+10x+1}{x^8-1000x^6-100x^4-10x^2-1} = \lim_{x\to\infty}\frac{\dfrac{10000}{x}+\dfrac{1000}{x^3}+\dfrac{100}{x^5}+\dfrac{10}{x^7}+\dfrac{1}{x^8}}{1-\dfrac{1000}{x^2}-\dfrac{100}{x^4}-\dfrac{10}{x^6}-\dfrac{1}{x^8}} = \frac{0}{1} = 0.$$

3. Implicit derivation gives $5y^2 + 10xyy' - 2e^{x+y}(1+y') + \dfrac{1}{3x+y+1}\cdot(3+y') = 0$. We take $x = 0$ and $y(0) = 0$ and we get $-2(1+y'(0)) + 3 + y'(0) = 0$, which implies $y'(0) = 1$.

Now we proceed with the second derivative of the expression:

$$10yy' + 10yy' + 10xy'y' + 10xyy'' -$$
$$-2e^{x+y}(1+y')^2 - 2e^{x+y}y'' + \frac{-1}{(3x+y+1)^2}(3+y')^2 + \frac{1}{3x+y+1}\cdot y'' = 0.$$

If we take $x = 0$, $y(0) = 0$ and $y'(0) = 1$ we get $-8 - 2y'' - 16 + y'' = 0$. This implies $y''(0) = -24$. The answer is $y'(0) + y''(0) = 1 - 24 = -23$.

4. We find out that $f'(x) = -6(x^2 - x - 2)$. The stationary points are the solutions to the equation $f'(x) = 0$, i.e. $x^2 - x - 2 = 0$: $x_1 = -1$ and $x_2 = 2$. The second derivative is $f''(x) = -6(2x-1)$. Since $f''(-1) = 18 > 0$ then $x_1 = -1$ local minimum point (and $f(-1) = -6$). Similarly we have $f''(2) = -18 < 0$, so $x_2 = 2$ is a local maximum point (and

$f(2) = 21$). Since the function is a polynomial, it is easy to find out that $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$ (we check the behavior of the term with highest power, i.e. $2x^3$). There is no global maximum, nor minimum.

5. $\frac{\partial f}{\partial x} = 3x^2y^2e^y + \frac{2}{2x-y} = 3x^2y^2e^y + 2(2x-y)^{-1}$ and

$\frac{\partial f}{\partial y} = x^3(2ye^y + y^2e^y) - \frac{1}{2x-y} = x^3(2y+y^2)e^y - (2x-y)^{-1}$. Now the second derivatives:

$\frac{\partial^2 f}{\partial x^2} = 6xy^2e^y - 4(2x-y)^{-2}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 3x^2(2y+y^2)e^y + 2(2x-y)^{-2}$ and

$\frac{\partial^2 f}{\partial y^2} = x^3\left[(2+2y)e^y + (2y+y^2)e^y\right] + (2x-y)^{-2} = x^3(2+4y+y^2)e^y - (2x-y)^{-2}$.

6. This is a geometric series with $r = \frac{1}{\sqrt{x-1}}$. It is obvious that x – 1 > 0 (The expression under the square root must be non-negative and moreover we cannot have zero in the denominator).

The series converges therefore for $0 < r < 1$. It means that $\sqrt{x-1} > 1$, i.e. $x-1>1$ Finally, $x > 2$.