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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Programme, October 21, 2009

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1. This geometric serie with $r = \frac{x^2+1}{5}$ is convergent if $-1 < r < 1$, i.e. only if $\frac{x^2+1}{5} < 1$. Thus $x^2 < 4$, and we have $-2 < x < 2$.
The sum equals $\frac{1}{1-\frac{x^2+1}{5}} = \frac{5}{4-x^2}$. The equation $\frac{5}{4-x^2} = 2$ implies $x = \pm\sqrt{3/2}$.

2. a) $\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{a\to 2^-} \int_0^a \frac{x}{\sqrt{4-x^2}}\,dx = ...$ substitution $4-x^2 = t$ gives $-2xdx = dz$ i.e. $xdx = -\frac{1}{2}dz$ and when x varies from 0 to a then t goes from 4 to $4-a^2$. Thus we have ... $\lim_{a\to 2^-} \int_4^{4-a^2} \frac{-\frac{1}{2}dz}{\sqrt{z}} = \lim_{a\to 2^-} -\int_4^{4-a^2} \frac{dz}{2\sqrt{z}} = \lim_{a\to 2^-} -\sqrt{z}\Big|_4^{4-a^2} = \lim_{a\to 2^-} -(\sqrt{4-a^2}-\sqrt{4}) = 2$. The integral converges.

b) $\int_0^\infty \frac{dt}{\sqrt{t+2009}} = \lim_{b\to\infty} \int_0^b \frac{dt}{\sqrt{t+2009}} = \lim_{b\to\infty} 2\sqrt{t+2009}\Big|_0^b = \lim_{b\to\infty} 2(\sqrt{b+2009}-\sqrt{2009})$. This integral diverges as b goes to infinity.

3. The point $(1,0)$ belongs to the curve (satisfies the equation). Differentiation of the expression (remember that y is a function of x, $y = y(x)$) gives $3x^2e^y + x^3e^yy' + 3y^2y'e^x + y^3e^x - 2xy^2 - 2x^2yy' + 1 + 2y' = 0$. Letting $x = 1$ and $y = y(1) = 0$ we get $3 + y' + 1 + 2y' = 0$ and thus $y' = -\frac{4}{3}$. The slope of the tangent line is then $y' = -\frac{4}{3}$ and the equation of the line is $y = -\frac{4}{3}x + b$. Since the point $(1,0)$ belongs to this line then we get $b = 1$ and the equation of the tangent line is $y = -\frac{4}{3}x + \frac{4}{3}$, i.e. $4x + 3y - 4 = 0$.

4. Since $f(2) = 3$ then $3 = f(2) = \frac{a\cdot 2^2 + b\cdot 2 + c}{2-1} = 4a + 2b + c$, which means that $4a + 2b + c = 3$. The derivative of $f(x)$ is $f'(x) = \frac{(2ax+b)(x-1)-(ax^2+bx+c)}{(x-1)^2} = \frac{ax^2-2ax-b-c}{(x-1)^2}$, and, since $f'(2) = 0$, then we get $-b-c = 0$. Having $c = -b$ we find (from the equation $a+b+c = -1$) that $a = -1$, and from the equation $4a+2b+c = 3$ we conclude that $b = 7$ and $c = -7$.
The function is thus $f(x) = \frac{-x^2+7x-7}{x-1}$ and and the derivative is $f'(x) = \frac{-x^2+2x}{(x-1)^2}$. We calculate now the second derivative: $f''(x) = \frac{(-2x+2)(x-1)^2 - 2(-x^2+2x)(x-1)}{(x-1)^4}$. Letting $x = 2$ we get $f''(2) = -2 < 0$. Thus the point in question is a local max.

5. Calculation of the determinants reduces the equation to $4x^2 + 4x - 7 = x^3 + x - 5$, which is the same as $x^3 - 4x^2 - 3x + 2 = 0$. We are first looking for possible rational solutions $\frac{p}{q}$, but since the coefficient at x^3 equals 1 then $q = 1$ and we will be looking for integer solutions.
The possible integers must be divisors of 2, hence we must check as x numbers 1, -1, 2 and -2. The control shows that $x = -1$ is a solution. Thus we can factorize (polynomial division) $x^3 - 4x^2 - 3x + 2 = (x+1)(x^2-5x+2)$ and what now remains is to solve the equation $x^2 - 5x + 2 = 0$. The two solutions are $x = \frac{5\pm\sqrt{17}}{2}$. The answer is then $x = -1$, $x = \frac{5+\sqrt{17}}{2}$ and $x = \frac{5-\sqrt{17}}{2}$.

6. The ln function is defined only for positive arguments, thus $1 - x^2 - y^2 > 0$. This gives the interior of the circle with radius 1 and centre at $(0,0)$ (the unit circle). However we are not supposed to have 0 in the denominator, hence $1 - x^2 - y^2 \neq 1$, i.e. $x^2 + y^2 \neq 0$. This excludes from the unit circle the centre point $(0,0)$.
When the points (x,y) approach the border of the unit circle from inside the circle (i.e. $x^2 + y^2 \to 1^-$) then $\ln(1 - (x^2 + y^2)) \to -\infty$ and then $g(x,y) \to 0^-$.
When the points (x,y) approach the centre $(0,0)$ of the unit circle (i.e. $x^2+y^2 \to 0$) then $\ln(1-(x^2+y^2)) \to 0^-$ and then $g(x,y) \to -\infty$.

7. We start by identifying the stationary points and thus we solve the equations $f'_x = 0$ and $f'_y = 0$, i.e. $6x^2 + 6xy = 6x(x+y) = 0$ and $-6y^2 + 3x^2 + 3 = 3(x^2 - 2y^2 + 1) = 0$.
The first equation implies that $x = 0$ or $y = -x$. If $x = 0$ the second equation gives $y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$.
If $y = -x$ the second equation gives $x^2 - 2x^2 + 1 = 0$, i.e. $x = \pm 1$. Thus we have four stationary points: $(0, \frac{\sqrt{2}}{2})$, $(0, -\frac{\sqrt{2}}{2})$, $(1,-1)$ and $(-1,1)$.
In order to find if the points are max-, min- or saddle-points we calculate: $A = f''_{xx} = 12x+6y$, $B = f''_{xy} = 6x$ and $C = f''_{yy} = -12y$. We look at the points in order:
$(0, \frac{\sqrt{2}}{2})$: $A = 3\sqrt{2}$, $B = 0$, $C = -6\sqrt{2}$ and $AC - B^2 = -36$. Thus a saddle-point.
$(0, -\frac{\sqrt{2}}{2})$: $A = -3\sqrt{2}$, $B = 0$, $C = 6\sqrt{2}$ and $AC - B^2 = -36$. Again a saddle-point.
$(1,-1)$: $A = 6$, $B = 6$, $C = 12$ and $AC - B^2 = 36$. A minimum point.
$(-1,1)$: $A = -6$, $B = -6$, $C = -12$ and $AC - B^2 = 36$. A maximum point.

Paul