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Solution of the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, January 11, 2010

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1. $\sum_{n=0}^{\infty} \frac{x^n + y^n}{(xy)^{n+2}} = \sum_{n=0}^{\infty} \frac{x^n}{(xy)^{n+2}} + \sum_{n=0}^{\infty} \frac{y^n}{(xy)^{n+2}} = \frac{1}{(xy)^2} \sum_{n=0}^{\infty} \left(\frac{1}{y}\right)^n + \frac{1}{(xy)^2} \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$. Since both geometric sums converge then $-1 < \frac{1}{x} < 1$ and $-1 < \frac{1}{y} < 1$ implying (don't forget that x and y are positive numbers) that $x > 1$ and $y > 1$. Hence those two inequalities together with $x + y < 3$ define the interior of the triangle in the plane with vertices at $(1, 1)$, $(1, 2)$ and $(2, 1)$.

2. a) Using twice integration by parts: $\int_1^e 1 \cdot (\ln x)^2\, dx = x(\ln x)^2 \Big|_1^e - \int_1^e x \cdot 2 \ln x \cdot \frac{1}{x}\, dx = e - 2 \int_1^e 1 \cdot \ln x\, dx = e - 2\Big(x \ln x \Big|_1^e - \int_1^e x \cdot \frac{1}{x}\, dx\Big) = e - 2\Big(e - x\Big|_1^e\Big) = e - 2(e - e + 1) = e - 2$.

b) $\int \frac{4x + 10}{\sqrt[3]{x^2 + 5x + 2}}\, dx = \ldots$ substitution $x^2 + 5x + 2 = z$ gives $(2x + 5)dx = dz$ and we have $\ldots = \int \frac{2(2x + 5)}{\sqrt[3]{x^2 + 5x + 2}}\, dx = \int \frac{2}{\sqrt[3]{z}}\, dz = 2 \int z^{-\frac{1}{3}}\, dz = 2 \cdot \frac{3}{2} \cdot z^{\frac{2}{3}} + C = 3(x^2 + 5x + 2)^{\frac{2}{3}} + C$.

3. The point $(1, -1)$ belongs to the curve (satisfies the equation). Differentiation of the expression (remember that y is a function of x, $y = y(x)$) gives $3x^2y^3 + 3x^3y^2y' - 4xy^2 - 4x^2yy' - 4y' + 2x = 0$. Letting $x = 1$ and $y = y(1) = -1$ we get $-3 + 3y' - 4 + 4y' - 4y' + 2 = 0$, which is the same as $3y' = 5$. Thus $y'(1) = \frac{5}{3}$. The slope of the tangent line is then $\frac{5}{3}$ and the equation of the line is $y = \frac{5}{3}x + b$. Since the point $(1, -1)$ belongs to this line then we get $b = -\frac{8}{3}$ and the equation of the tangent line is $y = \frac{5}{3}x - \frac{8}{3}$ or $5x - 3y - 8 = 0$.

4. The slope of the line $y = 12x + 5$ equals 12. Thus $f'(a) = 6a^2 - 6a = 12$ which implies that $a^2 - a - 2 = 0$. Solving this equation we get $a = -1$ and $a = 2$. For $x = a = -1$ we have on the line the point $(-1, -7)$ while on the graph of the function we have the point $(-1, 20)$. This must however be the same point (the point of tangency) so we reject $a = -1$. For $x = a = 2$ we have on the line and on the graph of the function the same point $(2, 29)$. The answer is thus $a = 2$.

5. After multiplying the matrix equality reduces to a linear system of equations

$$\begin{cases} 3x + 5y - 2z = 1 \\ -x + 3y + 5z = 0 \\ 4x - 2y - 5z = -1 \end{cases}$$

The determinant of the coefficients matrix equals 80 and using Cramer's rule (counting another three determinants) we find that $x = \frac{-36}{80} = -\frac{9}{20}$, $y = \frac{28}{80} = \frac{7}{20}$ and $z = \frac{-24}{80} = -\frac{3}{10}$.

6. The function is well defined for all real numbers x. For convenience let us write X instead of the expression $e^{\frac{x^2-10x}{12}}$. Now we find that $f'(x) = 6X + 6xX \cdot \frac{x - 5}{6} = X(6 + x^2 - 5x) = X(x - 2)(x - 3)$. Since the expression X is always positive then we have two stationary points $x = 2$ and $x = 3$. The study of signs prove that $f'(x) > 0$ for $x < 2$ and for $x > 3$. These are thus the intervals where the function is increasing. In the interval $2 < x < 3$ the derivative is negative, hence the function is decreasing. The conclusion is then that the function has a local max at $x = 2$ and local min at $x = 3$.
The graph crosses the y-axis ($x = 0$) at the point $(0, 0)$ and the x-axis ($y = 0$) only at the same point.

Finally the function has no global max nor global min since $\lim_{x\to\pm\infty} \dfrac{x^2-10x}{12} = \infty$ and thus $\lim_{x\to-\infty} 6xe^{\frac{x^2-10x}{12}} = -\infty$ and $\lim_{x\to\infty} 6xe^{\frac{x^2-10x}{12}} = \infty$.

7. 1. The stationary points inside the triangle: solving the equations $f'_x(x,y) = 0$ and $f'_y(x,y) = 0$, i.e. $y - 1 = 0$ and $x - 1 = 0$ we get the point $(x,y) = (1,1)$ which in fact is inside the triangle.

2. The boundary consists of three segments (the sides of the triangle): a) $y = 0$ and $0 < x < 2$, b) $x = 0$ and $0 < y < 4$, c) line segment $y = -2x + 4$, $0 < x < 2$ (segment between vertices $(2,0)$ and $(0,4)$). Each one of those cases we need to study separately.
a) $y = 0$ and $0 < x < 2$. The function becomes a function of one variable $g_1(x) = f(x,0) = -x + 3$. We find that $g'_1(x) = -1 \neq 0$. Hence there are no stationary points in this case.
b) $x = 0$ and $0 < y < 4$. The function becomes again a function of one variable $g_2(y) = f(0,y) = -y + 3$. Since $g'_2(y) = -1 \neq 0$ then there are no new stationary points.
c) $y = -2x + 4$, $0 < x < 2$. The new function is $g_3(x) = f(x, -2x+4) = x(-2x+4) - x - (-2x+4) + 3 = -2x^2 + 5x - 1$. Differentiation gives $g'_3(x) = -4x + 5$ and we have the equation $g'_3(x) = -4x + 5 = 0$. Hence $x = 5/4$ gives a new stationary point, $(x,y) = (5/4, 3/2)$, since $y = -2x + 4$.

3. The vertices of the triangle must be treated separately. The calculation gives finally: $f(0,0) = 3$, $f(2,0) = 1$, $f(0,4) = -1$, $f(1,1) = 2$ and $f(5/4, 3/2) = 17/8$. The comparison between those values implies that $f(0,0) = 3$ is a max value and that $f(0,4) = -1$ is a min value.

Paul