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Solution of the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, December 11, 2010

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1. This geometric series with $r = (1+x)^2$ is convergent if $-1 < r < 1$, i.e. only if $0 \le (1+x)^2 < 1$. This inequality reduces to $x(x+2) < 0$, implying $-2 < x < 0$.

2. a) $\int_0^1 (x^3+x)\ln(x^2+1)^2\,dx = \int_0^1 2x(x^2+1)\ln(x^2+1)\,dx = ...$ by substitution $u = x^2+1$, $du = 2x\,dx$ and u varying between 1 and 2, ... $= \int_1^2 u\ln u\,du = ...$ integration by parts ... $= \frac{u^2}{2}\ln u\Big|_1^2 - \int_1^2 \frac{u^2}{2}\frac{1}{u}du = 2\ln 2 - \frac{1}{2}\int_1^2 u\,du = 2\ln 2 - \frac{1}{4}u^2\Big|_1^2 = 2\ln 2 - \frac{3}{4}$.

b) $\int t^2\sqrt{t+1}\,dt$... substitution $s = t+1$, $ds = dt$ and $t^2 = (s-1)^2$ gives ... $\int (s-1)^2\sqrt{s}\,ds = \int (s^{5/2} - 2s^{3/2} + s^{1/2})\,ds = \frac{2}{7}s^{7/2} - 2\cdot\frac{2}{5}s^{5/2} + \frac{2}{3}s^{3/2} + C = \frac{2}{7}(t+1)^{7/2} - \frac{4}{5}(t+1)^{5/2} + \frac{2}{3}(t+1)^{3/2} + C$.

3. Inserting $x = 0$ into $xy^2(x) + y(x)\ln(x+y(x)) = e^x - 1$ gives $y(0)\cdot\ln(y(0)) = 0$. Since $y(0)$ must be a positive number then $y(0) = 1$.
The differentiation of the expression gives $y^2 + 2xyy' + y'\ln(x+y) + \frac{y}{x+y}\cdot(1+y') = e^x$. Inserting $x = 0$ and $y = y(0) = 1$ implies $1 + 1 + y' = 1$, which means that $y'(0) = -1$. The equation of the tangent line l can thus be written as $y = -x + m$ for some constant m. Since point $(0,1)$ lies on this line then $m = 1$ and the equation of l is $y = -x + 1$.
We need to find point(s) where the line l meets the parabola $y = x^2 - 2x - 1$. Inserting $y = -x + 1$ into $y = x^2 - 2x - 1$ we get $-x + 1 = x^2 - 2x - 1$, i.e. $x^2 - x - 2 = 0$. The roots are $x = -1$ and $x = 2$. Hence the points we were looking for are $(-1, 2)$ and $(2, -1)$.

4. Eveluating the three determinants gives the equation $x^3 + 2x^2 - 9x - 4 = 0$. Testing for the possible rational solutions (± 1, ± 2, ± 4) we find that $x = -4$ is a solution. Dividning by $x+4$ we get $x^3+2x^2-9x-4 = (x+4)(x^2-2x-1) = 0$. It remains to solve the quadratic $x^2 - 2x - 1 = 0$ and we have the three answers to the problem: $x = -4$, $x = 1 - \sqrt{2}$ and $x = 1 + \sqrt{2}$.

5. The function is well defined and continuos for all real x. The derivative, $f'(x) = (x^2 + 2x - 3)e^x$, is 0 if and only if $x = -3$ or $x = 1$ (the roots of the equation $x^2 + 2x - 3 = 0$). The study of the sign of the derivative shows that $f'(x) = (x+3)(x-1)e^x$ is negative only for $-3 < x < 1$. Thus $f(x)$ is increasing for $x < -3$ and for $x > 1$. Inbetween, the function $f(x)$ is decreasing. The point $x = -3$ is a local max point while $x = 1$ is a local minimum point, $f(-3) = 6e^{-3}$ and $f(1) = -2e$.
The second derivative $f''(x) = (x^2+4x-1)e^x$ can be factorized as $f''(x) = (x-(-2-\sqrt{5}))(x-(-2+\sqrt{5}))e^x$. The study of the sign shows that the function is concave for $x < -2-\sqrt{5}$ and for $x > -2+\sqrt{5}$, while it is convex for $-2-\sqrt{5} < x < -2+\sqrt{5}$.
Finally, $\lim_{x\to-\infty} f(x) = 0^+$ and $\lim_{x\to\infty} f(x) = \infty$. The function has no global maximum while x is a global minimum.

6. Partial derivatives are $f'_x = y^2 - 2xy$ and $f'_y = 2xy - x^2$. The system of equations $y^2 - 2xy = 0$ and $2xy - x^2 = 0$ gives, within the square $(0 < x < 1,\ 0 < y < 1)$, the equalities $y(y-2x) = 0$ and $x(2y-x) = 0$, i.e. $y(y-2x) = 0$ and $x(2y-x) = 0$. Now, the system $y = 2x$ and $x = 2y$ leads to $y = 4y$ which has no solutions within the square.

The boundary of the square consists of four line segments: $l_1: \ y=0$ and $0<x<1$, $\ l_2: \ x=0$ and $0<y<1$, $\ l_3: \ y=1$ and $0<x<1$ and $\ l_4: \ x=1$ and $0<y<1$. On the first two segments the value of the function is 0. On l_3 we have $f(x,1)=x-x^2$, which is a function of one variable x. The derivative of $g(x)=x-x^2$ is $g'(x)=1-2x$ which is 0 only if $x=\dfrac{1}{2}$. Hence we have one possible extremum point $P_1=(\dfrac{1}{2},1)$. Similarly we get the point $P_2=(1,\dfrac{1}{2})$ on l_4.
We check the value of $f(x,y)$ at the points P_1, P_2 and at the four vertices of the square: $f(\dfrac{1}{2},1)=\dfrac{1}{4}$, $f(1,\dfrac{1}{2})=-\dfrac{1}{4}$ and, for each vertex P we have $f(P)=0$. Hence the largest value of $f(x,y)$ is $\dfrac{1}{4}$ while the smallest is $-\dfrac{1}{4}$.

7. Calculating partial derivatives we find that $f'_x=\dfrac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}-\dfrac{1}{2}x^{-1}y$, $f'_y=\dfrac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}-\dfrac{1}{2}(\ln x+\ln y)-\dfrac{1}{2}$, $f''_{xx}=-\dfrac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}}+\dfrac{1}{2}x^{-2}y$, $f''_{yy}=-\dfrac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{2}}-\dfrac{1}{2}y^{-1}$ and $f''_{xy}=\dfrac{1}{4}x^{-\frac{1}{2}}y^{-\frac{1}{2}}-\dfrac{1}{2}x^{-1}$.
This gives us $x^2f''_{xx}-y^2f''_{yy}+2xyf''_{xy}=-\dfrac{1}{4}x^{\frac{1}{2}}y^{\frac{1}{2}}+\dfrac{1}{2}y-\dfrac{1}{4}x^{\frac{1}{2}}y^{\frac{1}{2}}+\dfrac{1}{2}y+\dfrac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}-y=\dfrac{1}{2}\sqrt{xy}$.

Paul