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**Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, October 22, 2011**

**1**. In order to find $a$ we need to find an integer solution to the equation $f'(x) = 0$, i.e. $f'(x) = 24x^3 - 120x^2 + 102x - 24 = 0$ (for this $x$ the tangent line will be parallel to $x-$axis). The equation can be reduced to $4x^3 - 20x^2 + 17x - 4 = 0$. The integer roots are among the numbers $\pm1, \pm2, \pm4$. By checking we find that $x = 4$ is a root. Thus $a = 4$.
Now, since $f''(x) = 72x^2 - 240x + 102$ then $f''(4) = 72 \cdot 16 - 240 \cdot 4 + 102 = 294 > 0$. Thus the function is convex at the point $x = 4$.

**2**. a) $\int (1+e^x)\ln(e^x+x)dx = ...$ by substitution $e^x+x = u$, $(e^x+1)dx = du$ $... = \int \ln u\, du = ...$ integration by parts $... = u\ln u - \int u\frac{1}{u}\,du = u\ln u - u + C = (e^x + x)\ln(e^x + x) \ - (e^x + x) \ + C.$

b) $\int_0^{0,5} \frac{1}{x\ln^2 x}\,dx = ...$ singularity at 0 $... = \lim_{b\to0^+} \int_b^{0,5} \frac{1}{x\ln^2 x}\,dx = ...$ substitution $\ln x = y$, $\frac{1}{x}dx = dy$ $... = \lim_{b\to0^+} \int_{\ln b}^{\ln 0,5} \frac{1}{y^2}\,dy = \lim_{b\to0^+} \Big(-\frac{1}{y}\Big)\Big|_{\ln b}^{\ln 0,5} = \lim_{b\to0^+} \Big(-\frac{1}{\ln 0,5} + \frac{1}{\ln b}\Big) = -\frac{1}{\ln 0,5} = \frac{1}{-\ln 0,5} = \frac{1}{\ln 2}.$

**3**. For $x = 0$ we have $y^3 + 8 = 0$ which implies $y = -2$. Differentiation gives $e^y y' x^2 + 2e^y x + e^x y^3 + 3e^x y^2 y' - 3x^2 = 4$. Letting $x = 0$ and $y = y(0) = -2$ we get $-8 + 12y' = 4$. Thus $y' = y'(0) = 1$. The equation of the tangent line $y = ax + b$ at the point $x = 0$ has the slope $a = 1$. Since this line passes through $(0, -2)$ then we have $-2 = 1 \cdot 0 + b$. Hence $b = -2$ and the answer is $y = x - 2$.

**4**. Finding stationary points by solving the system of equations: $f'_x = 3x^2 - 12 = 0$ and $f'_y = 3y^2 + 6y - 9 = 0$ gives $x = \pm2$ and $y_1 = 1$, $y_2 = -3$. Hence we have four points: $P_1 = (-2, -3)$, $P_2 = (-2, 1)$, $P_3 = (2, -3)$ and $P_4 = (2, 1)$. Then we have $f''_{xx} = 6x$, $f''_{xy} = 0$, $f''_{yy} = 6y - 6$.
Letting now $A = f''_{xx}(P)$, $B = f''_{xy}(P)$, $C = f''_{yy}(P)$, and $\Delta(P) = AC - B^2$ we get:
for $P_1$: $A = -12,\ B = 0,\ C = -12, \Delta = 144$, thus $P_1$ is a local maximum,
for $P_2$: $A = -12,\ B = 0,\ C = 12, \Delta = -144$, hence $P_1$ is a saddlepoint,
for $P_3$: $A = 12,\ B = 0,\ C = -12, \Delta = -144$, thus $P_1$ is a saddlepoint,
for $P_4$: $A = 12,\ B = 0,\ C = 12, \Delta = 144$, thus $P_1$ is a local minimum.

**5**. Before differentiation we may simplify the expression: $h(x) = \frac{1}{2}\ln(x^2 + y^2)$. Now, $h'_x = \frac{x}{x^2 + y^2}$ and $h''_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$. Similarly, $h'_y = \frac{y}{x^2 + y^2}$ and $h''_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$. Thus $h''_{xx} + h''_{yy} = 0$.
Since $\frac{\partial^2 h}{\partial x \partial y} = -\frac{2xy}{(x^2 + y^2)^2}$ then $-\frac{(x^2 + y^2)^2}{2}\frac{\partial^2 h}{\partial x \partial y} = xy.$

**6**. The quotient equals $r = \frac{2}{x^2 + 1}$ so the series converges when $-1 < \frac{2}{x^2 + 1} < 1$. Since $\frac{2}{x^2 + 1}$ is never negative then we have $0 \le \frac{2}{x^2 + 1} < 1$, i.e. $2 < x^2 + 1$. Thus $1 < x^2$. This implies that $x < -1$ or $x > 1$.

The sum expresses as $\frac{1}{1-\frac{2}{x^2+1}} = \frac{x^2+1}{x^2+1-2} = \frac{x^2+1}{x^2-1}$. Thus we need to solve the equation $\frac{x^2+1}{x^2-1} = 3$, i.e. $x^2+1 = 3x^2-3$. This reduces to $x^2 = 2$, and we have $x = \pm\sqrt{2}$.

**7**. The slope of a tangent line $y = ax + b$ is the number $a$ and equals the derivative $f'(x) = \frac{-12x}{(x^2+3)^2}$ at the given point. Thus we need to find the maximum and the minimum value of $g(x) = \frac{-12x}{(x^2+3)^2}$. Since $g'(x) = \frac{-36(1-x)(1+x)}{(x^2+3)^3}$ then it zeroes are where $(1-x)(1+x) = 0$, i.e. when $x = \pm 1$. Studying the sign chart of $g'(x)$ we find the $g'(x)$ is positive when $x$ is less than $-1$ or greater than 1 and is negative when $-1 < x < 1$.
Hence $g(x)$ has a local maximum at $x = -1$ and its value is $g(-1) = \frac{3}{4}$. The tangent line is then $y = \frac{3}{4}x + b$. This line passes through the point $\big(-1, f(-1)\big) = (-1, \frac{3}{2})$ and the equation $\frac{3}{2} = \frac{3}{4}\cdot(-1) + b$ gives $b = \frac{9}{4}$. Thus the tangent line we are looking for is $y = \frac{3}{4}x + \frac{9}{4}$.
Moreover $g(x)$ has a local minimum at $x = 1$ and its value is $g(1) = -\frac{3}{4}$. The tangent line is then $y = -\frac{3}{4}x + b$. This line passes through the point $\big(1, f(1)\big) = (1, \frac{3}{2})$ and the equation $\frac{3}{2} = -\frac{3}{4}\cdot(1) + b$ gives $b = \frac{9}{4}$. Thus the second tangent line we are looking for is $y = -\frac{3}{4}x + \frac{9}{4}$.

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