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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, December 17, 2011

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1. For $x = -1$ we have the point $(-1, -3 \cdot (-1) + 2) = (-1, 5)$ on the line and on the curve. Thus $(-1, 5)$ satisfies the equation, i.e. (1): $5 = -a + b + c$. Since $f'(x) = 3ax^2 + 2bx$ and $f'(-1) = 3a - 2b$ then we have the equation (2): $3a - 2b = -3$, (the slope of the tangent line). Finally $f''(x) = 6ax + 2b$ and, since $x = 1$ ia an inflexionspoint, then $f''(1) = 0$, i.e. (3): $6a + 2b = 0$. Solving the system of linear equations (1), (2) and (3) gives $a = -\frac{1}{3}$, $b = 1$ och $c = \frac{11}{3}$.

2. a) $\int \frac{\ln x\sqrt{\ln x + 1}}{x}\,dx = ...$ substitution $\ln x = u$, $\frac{1}{x}dx = du$ gives $... = \int u\sqrt{u+1}, du = ...$ substitution $v = u + 1$, $dv = du$ $... = \int (v-1)\sqrt{v}dv = \int (v-1)v^{\frac{1}{2}}dv = \int (v^{\frac{3}{2}} - v^{\frac{1}{2}})dv = \frac{2}{5}v^{\frac{5}{2}} - \frac{2}{3}v^{\frac{3}{2}} + C = \frac{2}{5}(u+1)^{\frac{5}{2}} - \frac{2}{3}(u+1)^{\frac{3}{2}} + C = \frac{2}{5}(\ln x + 1)^{\frac{5}{2}} - \frac{2}{3}(\ln x + 1)^{\frac{3}{2}} + C.$

b) $\int_0^\infty \frac{e^x}{\sqrt{e^x + 2012}}\,dx = ...$ singularity at infinity $... = \lim_{b\to\infty}\int_0^b \frac{e^x}{\sqrt{e^x + 2012}}\,dx = ...$ substitution $e^x + 2012 = t$, $e^x dx = dt$ $... = \lim_{b\to\infty}\int_{2013}^{e^{b+2012}} \frac{1}{\sqrt{t}}\,dt = \lim_{b\to\infty} 2\sqrt{t}\Big|_{2013}^{e^{b+2012}} = \lim_{b\to\infty} 2\sqrt{e^{b+2012}} - 2\sqrt{2013}$. The integral diverges.

3. Differentiation gives $2x\ln(xy) + x^2\frac{1}{xy}(y + xy') = e^{x^2+y^2-2}(2x + 2yy')$. Letting $x = y = 1$ we get $1 + y' = 2 + 2y'$. This implies $y' = -1$. The equation of the tangent line is then $y = -x + b$. Since this line passes through $(1, 1)$ then $1 = -1 + b$, and we get $b = 2$. The line $y = -x + 2$ cuts the axis at $(0, 2)$ and $(2, 0)$.

4. Since $A - \lambda E = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{pmatrix}$ then we need to solve the equation $\begin{vmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{vmatrix} = 0$. Calculating the determinant we get $-\lambda^3 - 5\lambda^2 - 3\lambda + 9 = 0$. All possible rational roots are among ± 1, ± 3, ± 9. We find that $\lambda = 1$ is one root and then, after dividing with $(\lambda - 1)$, we get $-\lambda^2 - 6\lambda - 9 = 0$, i.e. $-(\lambda + 3)^2 = 0$. The answer is $\lambda = 1$ and $\lambda = -3$ (a double root).

5. The function is well defined for all real x. The derivative is $f'(x) = 2(-1 + 3x - 2x^2)e^{x^2} = -2(x-1)(x - \frac{1}{2})e^{x^2}$. The derivative is zero only if $x = 1$ and $x = \frac{1}{2}$. If we look at the sign chart for $f'(x)$ we find that $f'(x)$ is negative for $x < \frac{1}{2}$ and for $x > 1$. For those x the function is decreasing. For $\frac{1}{2} < x < 1$ the derivative is positive and thus the function is increasing. Consequently at $x = \frac{1}{2}$ we have a local minimum and at $x = 1$ we have a local maximum. The values are $f(\frac{1}{2}) = 2e^{\frac{1}{4}}$ and $f(1) = e$.
It is quite obvious that $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$. Thus there are no global maximum nor minimum. Finally, the intersection with $x-$axis is when $y = 0$, i.e. $(3 - 2x)e^{x^2} = 0$. This happens only when $x = \frac{3}{2}$. The intersection with $y-$axis is when $x = 0$ i.e. $y = 3$.

6. $g'_x = e^y + ye^x$, $g''_{xx} = ye^x$ and $g'''_{xxx} = ye^x$. Similarly, $g'_y = xe^y + e^x$, $g''_{yy} = xe^y$ and $g'''_{yyy} = xe^y$. For mixed derivatives we have $g'''_{xxy} = e^x$ and $g'''_{xyy} = e^y$. Hence $\frac{\partial^3 g}{\partial x^3} + \frac{\partial^3 g}{\partial y^3} - x\frac{\partial^3 g}{\partial x \partial y^2} - y\frac{\partial^3 g}{\partial x^2 \partial y} = 0$.

7. The triangle bounds by the lines $l_1: \ y = 0$, $l_2: \ x = 0$ and $l_3: \ y = -\frac{3}{4}x + 3$. The points within the triangle satisfy the inequalities $x \geq 0$, $y \geq 0$ and $y \leq 3 - \frac{3}{4}x$.

First, we look for critical points inside the triangle. We have $f'_x = 2x - 4$ and $f'_y = 6y - 6$. We see that there is only one critical point, at $(x_0, y_0) = (2, 1)$. The point $P_1 = (2, 1)$ satisfies all of inequalities defining the triangle; it is therefore a candidate for an absolute maximum or minimum.

We now check the boundary, by examining each edge of the triangle individually. On $l_1: \ y = 0, \ 0 < x < 4$ we have $y = 0$, which yields $f(x, 0) = x^2 - 4x = g_1(x)$ - a function of one variable. We then have $g'_1(x) = 2x - 4$ which has a critical point at $x = 2$. Therefore, $P_2 = (2, 0)$ is a candidate for an absolute extremum.

On $l_2: \ x = 0, \ 0 < y < 3$ we have $x = 0$, which yields $f(0, y) = 3y^2 - 6y = g_2(y)$ - again a function of one variable. We then have $g'_2(y) = 6y - 6$ which has a critical point at $y = 1$. Hence, $P_3 = (0, 1)$ is a new candidate for an absolute extremum.

On $l_3: \ y = -\frac{3}{4}x + 3, \ 0 < x < 4$, we have $f(x, -\frac{3}{4}x + 3) = \frac{43}{16}x^2 - 13x + 9 = g_3(x)$. We get then $g'_3(x) = \frac{43}{8}x - 13$ which has a critical point at $x = \frac{13}{\frac{43}{8}} = \frac{104}{43}$. Since then $y = -\frac{3}{4} \cdot \frac{104}{43} + 3 = \frac{51}{43}$ then $P_4 = (\frac{104}{43}, \frac{51}{43})$ is also a candidate for an absolute extremum.

Finally, we must include the vertices of the triangle, $P_5 = (4, 0)$, $P_6 = (0, 3)$ and $P_7 = (0, 0)$.

Now we calculate $f(P_1) = -7$, $f(P_2) = -4$, $f(P_3) = -1$, $f(P_4) = -\frac{289}{43}$, $f(P_5) = 0$, $f(P_6) = 9$ and $f(P_7) = 0$. We conclude that the minimum value of the function, -7, is at $(2, 1)$, and the maximum value, 9, is at $(0, 3)$.

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