Department of Mathematics
STOCKHOLM UNIVERSITY

Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, August 21, 2012

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1. Differentiating the expression gives $3(x+1)^2y(x)^2+2(x+1)^3y(x)y'(x)+\dfrac{1}{x+1}\ln y(x)+\ln(x+1)\dfrac{1}{y(x)}y'(x)+e^{xy(x)}\big(y(x)+xy'(x)\big)=0$. Letting in $x=0$ and $y(0)=1$ we get $3+2y'(0)+1=0$, so $y'(0)=-2$. The slope of the tangent line is then -1 and the equation of the line is $y=-2x+m$. Since this line passes through the point $(x,y)=(0,1)$, we get $m=1$. The line $y=-2x+1$ crosses the x-axis ($y=0$) at the point $x=\dfrac{1}{2}$.

2. a) Since $\lim_{x\to 5}(x^3-5x)=100$ and the denominator goes to 0, the limit either is $+\infty$ or $-\infty$ or does not exist. Now, the denominator may go to 0^+ (when $x\to 5^-$) or to 0^- (when $x\to 5^+$). Thus the limit does not exist.

b) $\lim_{x\to 2}\left(\dfrac{5x+2}{x^2-4}-\dfrac{3}{x-2}\right)=\lim_{x\to 2}\left(\dfrac{5x+2}{x^2-4}-\dfrac{3(x+2)}{(x+2)(x-2)}\right)=\lim_{x\to 2}\left(\dfrac{5x+2}{x^2-4}-\dfrac{3(x+2)}{x^2-4}\right)=\lim_{x\to 2}\dfrac{2(x-2)}{x^2-4}=\lim_{x\to 2}\dfrac{2}{x+2}=\dfrac{1}{2}$.

c) $\lim_{x\to 0}\dfrac{1+2x-e^{2x}}{x\ln(x+1)}=$ applying de l'Hôpial's rule $=\lim_{x\to 0}\dfrac{2-2e^{2x}}{\ln(x+1)+\frac{x}{x+1}}=$ applying de l'Hôpial's rule again $=\lim_{x\to 0}\dfrac{-4e^{2x}}{\frac{1}{x+1}+\frac{1}{(x+1)^2}}=\dfrac{-4}{2}=-2$.

3. Since $f(2)=3$ then $3=f(2)=\dfrac{a\cdot 2^2+b\cdot 2+1}{2-1}=4a+2b+1$, which means that $2a+b=1$. The derivative of $f(x)$ is $f'(x)=\dfrac{(2ax+b)(x-1)-(ax^2+bx+1)}{(x-1)^2}=\dfrac{ax^2-2ax-b-1}{(x-1)^2}$, and, since $f'(2)=0$, we get $-b-1=0$. Having $b=-1$ we find (from the equation $2a+b=1$) that $a=1$.
The function is thus $f(x)=\dfrac{x^2-x+1}{x-1}$ and we calculate the second derivative:
$f''(x)=\dfrac{(2x-2)(x-1)^2-2(x^2-2x)(x-1)}{(x-1)^4}$. Letting $x=2$ we get $f''(2)=2>0$. Thus the point in question is a local minimum.

4. In order not to repeat ourselves, let's write X instead for the expression $e^{-(\frac{1}{2}x^2+y^2+2y)}$. The first partial derivatives are then $f'_x=X+xX(-x)=(1-x^2)X$ and $f'_y=xX(-2y-2)=-2x(y+1)X$. Since X is always positive then, solving the system of equations $f'_x=f'_y=0$, we get two points: $P_1=(1,-1)$ and $P_2=(-1,-1)$.

Now we calculate the second derivatives: $f''_{xx}=-2xX+(1-x^2)X(-x)=(x^3-3x)X$, $f''_{xy}=(1-x^2)X(-2y-2)=-2(1-x^2)(y+1)X$ and $f''_{yy}=-2xX-2x(y+1)X(-2y-2)=-2xX+4x(y+1)^2X=2x(2y^2+4y+1)X$.

For the point P_1 we find out that $A=f''_{xx}(P_1)=(1-3)X=-2X$, and, since X is always positive, $A<0$. $B=f''_{xy}(P_1)=0$ and $C=f''_{yy}(P_1)=-2X$. Thus $AC-B^2=4X^2>0$, hence P_1 is a local maximum point. Similarly, for the point P_2 we find out that $A=f''_{xx}(P_2)=2X>0$, $B=f''_{xy}(P_2)=0$ and $C=f''_{yy}(P_2)=2X$. Thus $AC-B^2=4X^2>0$, hence P_2 is a local minimum point.

5. a) The standard procedure (calculating the determinants) gives $x=\dfrac{-4}{-4}=1$, $y=\dfrac{-4}{-12}=3$ and $z=\dfrac{-8}{-4}=2$.

b) The equation can be written as $A \cdot X = C - B$ and, since X must be a 2x2 matrix, then taking $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ we get $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 1 & 1 \end{pmatrix}$.
This leads us to a system of four equations: $2x + 3z = 7, \quad x - z = 1. \ 2y + 3t = 2, \quad y - t = 1$. The solution is $x = 2, \quad z = 1. \ y = 1, \quad t = 0$. Hence $X = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$.

6. This geometric serie with $r = \dfrac{2}{3} \ln x$ is convergent if $-1 < r < 1$, i.e. $-1 < \dfrac{2}{3} \ln x < 1$. Thus $-\dfrac{3}{2} < \ln x < \dfrac{3}{2}$, and we have $e^{-\frac{3}{2}} < x < e^{\frac{3}{2}}$.
The sum S equals $\dfrac{1}{1 - \frac{2}{3} \ln x} = \dfrac{3}{3 - 2\ln x}$, hence, if $S = \frac{3}{2}$ then $\dfrac{3}{3 - 2\ln x} = \dfrac{3}{2}$. This gives $\ln x = \dfrac{1}{2}$ and then $x = \sqrt{e}$.

7. a) Certainly $x^2 - 1 > 0$, since the logarithm is only defined for positive real numbers. Thus $x < -1$ or $x > 1$. At the same time $x^2 - 1 \neq 1$; otherwise we would have 0 in the denominator. Thus $x \neq \pm\sqrt{2}$. Hence $f(x)$ is defined in the intervals $(-\infty, -\sqrt{2}), \ (-\sqrt{2}, -1), \ (1, \sqrt{2})$ and $(\sqrt{2}, \infty)$.
b) The derivative $f'(x) = -\dfrac{1}{\big(\ln(x^2-1)\big)^2} \cdot \dfrac{2x}{x^2 - 1}$. The first factor is always negative and $x^2 - 1$ is always positive (for those x for which the function is defined). Thus the function is increasing if $x < 0$ and decreasing if $x > 0$. Hence the function is increasing in the intervals $(-\infty, -\sqrt{2})$ and $(-\sqrt{2}, -1)$, and decreasing in the intervals $(1, \sqrt{2})$ and $(\sqrt{2}, \infty)$.

Paul