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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, November 3, 2012

1. This geometric series with $r = \frac{x^2-1}{3}$ is convergent if $-1 < r < 1$, i.e. only if $\frac{x^2-1}{3} < 1$. Thus $x^2 < 4$, and we have $-2 < x < 2$.
The sum equals $\frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{4-x^2}$. The equation $\frac{3}{4-x^2} = 3$ implies $x = \pm\sqrt{3}$.

2. a) $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx = \lim_{a\to 3^-} \int_0^a \frac{x}{\sqrt{9-x^2}}\,dx = ...$ substitution $9-x^2 = s$ gives $-2xdx = ds$ i.e. $xdx = -\frac{1}{2}ds$ and when x varies from 0 to a then s goes from 9 to $9-a^2$. Thus we have ... $\lim_{a\to 3^-} \int_9^{9-a^2} \frac{-\frac{1}{2}ds}{\sqrt{s}} = \lim_{a\to 3^-} -\int_9^{9-a^2} \frac{ds}{2\sqrt{s}} = \lim_{a\to 3^-} -\sqrt{s}\Big|_9^{9-a^2} = \lim_{a\to 3^-} -(\sqrt{9-a^2}-\sqrt{9}) = 3$. The integral converges.

b) $\int_0^\infty \frac{dz}{\sqrt{z+1}} = \lim_{b\to\infty} \int_0^b \frac{dz}{\sqrt{z+1}} = \lim_{b\to\infty} 2\sqrt{z+1}\Big|_0^b = \lim_{b\to\infty} 2(\sqrt{b+1}-\sqrt{1})$. This integral diverges as b goes to infinity.

3. The point $(1,0)$ belongs to the curve (satisfies the equation). The expression can be rewritten as: $2x^3e^y - y^3e^x - \ln x - \ln(y+1) + x + y - 3 = 0$. Differentiation of the expression (remember that y is a function of x, $y = y(x)$) gives $6x^2e^y + 2x^3e^yy' - 3y^2y'e^x - y^3e^x - \frac{1}{x} - \frac{y'}{y+1} + 1 + y' = 0$.
Letting $x = 1$ and $y = y(1) = 0$ we get $6 + 2y' = 0$ and thus $y' = -3$. The slope of the tangent line is then $y' = -3$ and the equation of the line is $y = -3x + b$. Since the point $(1,0)$ belongs to this line then we get $b = 3$ and the equation of the tangent line is $y = -3x + 3$, i.e. $3x + y - 3 = 0$.

4. Since $f(1) = 7$ then $7 = f(1) = \frac{a+b-3b}{2}$, which means that $a - 2b = 14$. The derivative of $f(x)$ is $f'(x) = \frac{(3ax^2+b)(x+1)-(ax^3+bx-3b)}{(x+1)^2} = \frac{2ax^3+3ax^2+4b}{(x+1)^2}$, and, since $f'(1) = 0$, then we get $5a + 4b = 0$.
Solving now the system of equations: $a - 2b = 14$ and $5a + 4b = 0$ we get $a = 4$ and $b = -5$.
The function is thus $f(x) = \frac{4x^3-5x+15}{x+1}$ and and the derivative is $f'(x) = \frac{8x^3+12x^2-20}{(x+1)^2}$. We calculate now the second derivative: $f''(x) = \frac{(24x^2+24x)(x+1)^2 - 2(8x^3+12x^2-20)(x+1)}{(x+1)^4}$. Letting $x = 1$ we get $f''(1) = 12 > 0$. Thus the point in question is a local min.

5. Calculation of the determinants reduces the equation to $-4z^2 - 4z + 7 = -z^3 - z + 5$, which is the same as $z^3 - 4z^2 - 3z + 2 = 0$. We are first looking for possible rational solutions $\frac{p}{q}$, but since the coefficient at z^3 equals 1 then $q = 1$ and we will be looking for integer solutions.
The possible integers must be divisors of 2, hence we must check as z numbers 1, -1, 2 and -2. The control shows that $z = -1$ is a solution. Thus we can factorize (polynomial division) $z^3 - 4z^2 - 3z + 2 = (z+1)(z^2 - 5z + 2)$ and what now remains is to solve the equation $z^2 - 5z + 2 = 0$. The two solutions are $z = \frac{5\pm\sqrt{17}}{2}$. The answer is then $z = -1$, $z = \frac{5+\sqrt{17}}{2}$ and $z = \frac{5-\sqrt{17}}{2}$.

6. The ln function is defined only for positive arguments, thus $1 - x > 0$ and $x > 0$. The same applies to the variable y. This gives all points (x, y) within the square $D = \{(x, y) : 0 < x < 1, 0 < y < 1\}$.
Since $\ln x$, $\ln y$, $\ln(1-x)$ and $\ln(1-y)$ are negative inside D then $g(x, y) > 0$ inside D. Thus the equation $g(x, y) = -2$ has no solutions. On the other hand, the equation $g(x, y) = 2$ has at least one solution, for example $x = y = \frac{1}{2}$.

7. We start by identifying the stationary points and thus we solve the equations $f'_x = 0$ and $f'_y = 0$, i.e. $6xy + 6x^2 = 6x(x + y) = 0$ and $-3 + 3x^2 + 6y^2 = 3(x^2 + 2y^2 - 1) = 0$.
The first equation implies that $x = 0$ or $y = -x$. If $x = 0$ the second equation gives $y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$.
If $y = -x$ the second equation gives $3y^2 - 1 = 0$, i.e. $y = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$. Thus we have four stationary points: $(0, \frac{\sqrt{2}}{2})$, $(0, -\frac{\sqrt{2}}{2})$, $(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3})$ and $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.
In order to find if the points are max-, min- or saddle-points we calculate: $A = f''_{xx} = 12x+6y$, $B = f''_{xy} = 6x$ and $C = f''_{yy} = 12y$. We look at the points in order:
$(0, \frac{\sqrt{2}}{2})$: $A = 3\sqrt{2}$, $B = 0$, $C = 6\sqrt{2}$ and $AC - B^2 = 36$. Thus a local minimum.
$(0, -\frac{\sqrt{2}}{2})$: $A = -3\sqrt{2}$, $B = 0$, $C = -6\sqrt{2}$ and $AC - B^2 = 36$. A local maximum point.
$(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3})$: $A = 2\sqrt{3}$, $B = 2\sqrt{3}$, $C = -4\sqrt{3}$ and $AC - B^2 = -36$. A saddle-point.
$(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$: $A = -2\sqrt{3}$, $B = -2\sqrt{3}$, $C = 4\sqrt{3}$ and $AC - B^2 = -36$. Again a saddle-point.

Paul