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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, December 15, 2012

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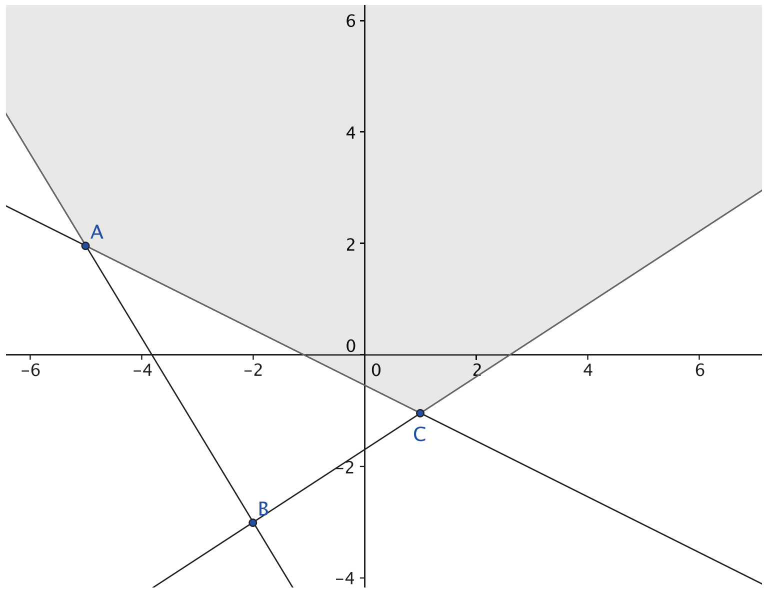
1. Each of the three equations: $x + 2y + 1 = 0$, $5x + 3y + 19 = 0$ and $-2x + 3y + 5 = 0$ defines a line in the plane. By solving a system for each of the pair of the equations we get the intersection points for those three pairs of lines: $A = (-5, 2)$, $B = (-2, -3)$ and $C = (1, -1)$. By studying which side of the line is defined by each of the inequality we find out the region the system describes.



2. a) Using twice integration by parts: $\int_1^2 x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2\Big|_1^2 - \int_1^2 \frac{1}{2}x^2 \cdot 2\ln x \cdot \frac{1}{x}\,dx = 2(\ln 2)^2 - \int_1^2 x\ln x\,dx = 2(\ln 2)^2 - \Big(\frac{1}{2}x^2 \ln x\Big|_1^2 - \frac{1}{2}\int_1^2 x^2 \cdot \frac{1}{x}\,dx\Big) = 2(\ln 2)^2 - \Big(2\ln 2 - \frac{1}{4}x^2\Big|_1^2\Big) = 2(\ln 2)^2 - 2\ln 2 + \frac{3}{4}$.

b) $\int \frac{e^x + 1}{(e^x + x + 1)^3}\,dx = ...$ substitution $e^x + x + 1 = z$ gives $(e^x + 1)dx = dz$ and we have $... = \int \frac{1}{z^3}\,dz = -\frac{1}{2}z^{-2} + C = -\frac{1}{2}(e^x + x + 1)^{-2} + C$.

3. Letting $x = x_0$, $y = 1$ into the equation $2x^3y^2 + 3x^2y^3 - x^2y + y^2 + xy = 0$ we get $2x_0^3 + 3x_0^2 - x_0^2 + 1 + x_0 = 0$, i.e. $2x_0^3 + 2x_0^2 + 1 + x_0 = 2x_0^2(x_0 + 1) + (x_0 + 1) = (x_0 + 1)(2x_0^2 + 1) = 0$. Since the parenthesis is never 0 we find that $x_0 = -1$. Thus the point of tangency is $(-1, 1)$.
Differentiation of the original expression (remember that y is a function of x, $y = y(x)$) gives $6x^2y^2 + 4x^3yy' + 6xy^3 + 9x^2y^2y' - 2xy - x^2y' + 2yy' + y + xy' = 0$.
Letting $x = -1$ and $y = y(1) = 1$ we get $6 - 4y' - 6 + 9y' + 2 - y' + 2y' + 1 - y' = 0$, which reduces to $3 + 5y' = 0$, and thus $y' = y'(1) = -\frac{3}{5}$, the slope of the tangent line.
The equation of the line is then $y = -\frac{3}{5}x + b$. Since the point $(-1, 1)$ belongs to this line then we get $b = \frac{2}{5}$ and the equation of the tangent line is $y = -\frac{3}{5}x + \frac{2}{5}$ or $3x + 5y - 2 = 0$.

4. Let's check if the equation has rational roots. The numerator must then be a divisor of 1 while the denominator must be a divisor of 4. The possible roots are ± 1, $\pm\frac{1}{2}$ and $\pm\frac{1}{4}$. By checking we find that $x = -\dfrac{1}{2}$ is a root. Dividing $4x^3 - 2x^2 - 4x - 1 = 0$ by $(x + \dfrac{1}{2})$ we can express our equation as $(x + \dfrac{1}{2})(4x^2 - 4x - 2) = 0$. In order to find the remaining two roots of the given equation we solve $4x^2 - 4x - 2 = 0$ getting $x = \dfrac{-1 \pm \sqrt{3}}{2}$. The answer is $x = -\dfrac{1}{2}$ and $x = \dfrac{-1 \pm \sqrt{3}}{2}$.

5. After multiplying the matrix equality reduces to a linear system of equations

$$\begin{cases} 3x + 5y - 2z = 1 \\ -x + 3y + 5z = 0 \\ 4x - 2y - 5z = -1 \end{cases}$$

The determinant of the coefficients matrix equals 80 and using Cramer's rule (after counting another three determinants) we find that $x = \dfrac{-36}{80} = -\dfrac{9}{20}$, $y = \dfrac{28}{80} = \dfrac{7}{20}$ and $z = \dfrac{-24}{80} = -\dfrac{3}{10}$.

6. The function is well defined for all real numbers x. For convenience let us write A instead of the expression e^{4x+1}. Now we find that $f'(x) = 4xA + (2x^2 - 1) \cdot 4A = 4A(2x^2 + x - 1) = 8A(x + 1)(x - \dfrac{1}{2})$. Since the expression A is always positive then we have two stationary points $x = -1$ and $x = \dfrac{1}{2}$ (points where $f'(x) = 0$).
A study of signs proves that $f'(x) > 0$ for $x < -1$ and for $x > \dfrac{1}{2}$. These are thus the intervals where the function is increasing. In the interval $-1 < x < \dfrac{1}{2}$ the derivative is negative, hence the function is decreasing. The conclusion is then that the function has a local max at $x = -1$ and local min at $x = \dfrac{1}{2}$.
The graph crosses the y-axis ($x = 0$) at the point $(0, -e)$ and the x-axis ($y = 0$) at the points where $2x^2 - 1 = 0$, i.e. at the points $(\pm\dfrac{\sqrt{2}}{2}, 0)$.
Finally, the function has no global max since $\lim\limits_{x\to\infty} f(x) = \infty$.
On the other hand $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (2x^2 - 1)e^{4x+1} = \lim\limits_{x\to-\infty} \dfrac{2x^2 - 1}{e^{-4x-1}}$ (the last step was to get the expression of the type $\dfrac{\infty}{\infty}$). Now, using the De L'Hopital's rule twice (!!) we get $\lim\limits_{x\to-\infty} \dfrac{2x^2 - 1}{e^{-4x-1}} = \lim\limits_{x\to-\infty} \dfrac{4x}{-4e^{-4x-1}} = \lim\limits_{x\to-\infty} \dfrac{4}{16e^{-4x-1}} = 0^+$. But, on the other hand, we have a local minimum at $x = \dfrac{1}{2}$ and the value there is $f(\dfrac{1}{2}) = -\dfrac{1}{2}e^3$, a negative number. Thus $x = \dfrac{1}{2}$ is the global minimum point of $f(x)$.

7. For convenience let us write B instead of the expression $e^{x^2 - \frac{1}{2}y^2}$. Thus $f(x, y) = yB$. The first partial derivatives are then $f'_x = 2xyB$ and $f'_y = B - y^2B = (1 - y^2)B$. Since B is always positive then, solving the system of equations $f'_x = f'_y = 0$, we get two points: $P_1 = (0, 1)$ and $P_2 = (0, -1)$.

Now we calculate the second derivatives: $f''_{xx} = 2yB + 4x^2yB = 2yB(1 + 2x^2)$, $f''_{xy} = 2xB + 2xyB(-y) = 2xB(1 - y^2)$ and $f''_{yy} = -2yB + (1 - y^2)B(-y) = -yB(3 - y^2)$.

For the point P_1 we find out that $A = f''_{xx}(P_1) = 2e^{-\frac{1}{2}}$, $B = f''_{xy}(P_1) = 0$ and $C = f''_{yy}(P_1) = -2e^{-\frac{1}{2}}$. Thus $AC - B^2 = -4e^{-1} < 0$, hence P_1 is a saddle point.
Similarly, for the point P_2 we find out that $A = f''_{xx}(P_2) = -2e^{-\frac{1}{2}}$, $B = f''_{xy}(P_2) = 0$ and $C = f''_{yy}(P_2) = 2e^{-\frac{1}{2}}$. Thus $AC - B^2 = -4e^{-1} < 0$, and also P_2 is a saddle point.

Paul