

**Time:** 09:00-14:00

**Instructions:**

- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Only **non-graphic** calculators are allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. JUSTIFY your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- Mark clearly where is your final answer putting A BOX around it.

**Grades:** Each solved problem is awarded by up to 10 points. At least 35 points are necessary for the grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

1. Calculate the limits

$$\text{a) } \lim_{x \rightarrow +\infty} \left( \frac{x^2}{x+2} - x \right), \quad \text{b) } \lim_{x \rightarrow 0^+} \frac{e^{x^2} - 1}{x^2 \ln x}.$$

**Solution** a) Observe that

$$\lim_{x \rightarrow +\infty} \left( \frac{x^2}{x+2} - x \right) = \lim_{x \rightarrow +\infty} \frac{-2x}{x+2} = \lim_{x \rightarrow +\infty} \frac{-2}{1+2/x} = \boxed{-2}.$$

b) We have an indetermination of the type  $0/0$ , so by L'Hôpital's rule

$$\lim_{x \rightarrow 0^+} \frac{1 - e^{x^2}}{x^2 \ln x} = \lim_{x \rightarrow 0^+} \frac{-2xe^{x^2}}{2x \ln x + x} = \lim_{x \rightarrow 0^+} \frac{-2e^{x^2}}{2 \ln x + 1} = \boxed{0}.$$

2. Consider the function  $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + \frac{1}{2}$ .

- Find the critical points;
- Find the intervals on which  $f$  is increasing and decreasing;
- Give the maximum and minimum value of  $f$  on the interval  $[0, 2]$ .

**Solution** (a) We have

$$f'(x) = x^2 + x - 2 = (x-1)(x+2),$$

so  $f'(x) = 0$  when  $x = 1$  and  $x = -2$ . So the critical points are

$$\boxed{(1, f(1)) = (1, -2/3) \text{ and } (-2, f(-2)) = (-2, 23/6)}.$$

(b)  $f'(x)$  is positive on  $(-\infty, -2)$  and  $(1, \infty)$ , hence  $f$  is increasing on these intervals.  $f'(x)$  is negative on  $(-2, 1)$ , hence  $f$  is decreasing on this interval.

(c) We calculate

$$f(0) = \frac{1}{2} \quad f(1) = -\frac{2}{3} \quad f(2) = \frac{7}{6}$$

so the minimum is at  $x = 1$  and the maximum is at  $x = 2$ . (Note that  $f(-2) = 23/6$  is *not* the maximum since  $-2 \notin [0, 2]$ )

3. The equation  $x^2y + 3y^3 = 7$  implicitly defines  $y$  as a function of  $x$ :  $y = y(x)$ .

(a) Find the value  $y(2)$ .

(b) Find the equation of the tangent line the curve at the point  $P = (2, y(2))$ .

**Solution** (a) Plugging in  $x = 2$ , we have

$$4y + 3y^3 = 7$$

or

$$3y^3 + 4y - 7 = 0.$$

Checking for integer solutions with  $y \in \{\pm 1, \pm 7\}$ , we see that  $y = 1$  is a solution.

(b) Differentiating implicitly gives

$$2xy + x^2y' + 9y^2y' = 0;$$

Plugging the point  $P = (2, 1)$  gives

$$4 + 4y' + 9y' = 0$$

and hence  $y'(2) = -\frac{4}{13}$ .

Hence an equation for the tangent line at this point is  $y = -\frac{4}{13}x + \frac{21}{13}$ .

4. Calculate the integrals

$$\text{a) } \int_0^2 \frac{x^2 + 4x + 2}{x + 1} dx; \quad \text{b) } \int x^3 e^{x^2} dx.$$

**Solution** a) To solve the integral we will first make a substitution  $u = x + 1$ , the integral then becomes

$$\int_1^3 \frac{(u-1)^2 + 4(u-1) + 2}{u} du = \int_1^3 \frac{u^2 + 2u - 1}{u} du = \int_1^3 \left(u + 2 - \frac{1}{u}\right) du.$$

If we integrate this we get

$$\left[\frac{1}{2}u^2 + 2u - \ln(u)\right]_1^3 = 8 - \ln(3).$$

So the answer is  $8 - \ln(3)$ .

b) To solve the integral we first make the substitution  $u = x^2$ , then we get that  $dx = \frac{1}{2x} du$  and the integral becomes

$$\int \frac{1}{2} u e^u du.$$

If we use integration by parts we get

$$\int \frac{1}{2} u e^u du = u e^u - e^u + C.$$

After substituting  $u = x^2$  the final answer becomes

$$\int x^3 e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C.$$

5. Find all the possible solutions of the following system of linear equations

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 5; \\x_1 + 2x_2 + 2x_3 + 2x_4 &= 10; \\x_1 + 2x_2 + 3x_3 + 3x_4 &= 15; \\x_1 + 2x_2 + 3x_3 + 4x_4 &= 20.\end{aligned}$$

First we will write the system of equations in matrix form and we get

$$\left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 2 & 10 \\ 1 & 2 & 3 & 3 & 15 \\ 1 & 2 & 3 & 4 & 20 \end{array} \right)$$

We will use Gaussian elimination to find the solutions. First we subtract the first row from the second, third and fourth row and we get

$$\left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 & 10 \\ 0 & 1 & 2 & 3 & 15 \end{array} \right)$$

Then we subtract the second row from the third and the fourth row and get

$$\left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 & 10 \end{array} \right)$$

After that we subtract the third row from the fourth row and get

$$\left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right)$$

From this we conclude that the solutions of our system of equations is  $x_4 = 5$  and  $x_1 = x_2 = x_3 = 0$ .

6. Compute the Taylor polynomial of degree 2 for the function  $g(x) = \sqrt{4 + 6x + 2x^2}$ , centered at the point  $a = 0$ .

**Solution** The solution is given by

$$P(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2$$

Now  $g'(x) = \frac{6+4x}{2\sqrt{4+6x+2x^2}}$  and  $g''(x) = -\frac{(6+4x)^2}{4(4+6x+2x^2)^{3/2}} + \frac{2}{\sqrt{4+6x+2x^2}}$ . Hence

$$g(0) = 2, \quad g'(0) = \frac{3}{2} \quad \text{and} \quad g''(0) = -\frac{1}{8}.$$

Thus  $P(x) = 2 + \frac{3}{2}x - \frac{1}{16}x^2$ .

7. Let  $F(x, y) = x^2 + xy + y^2 + x + 2y + 14$ . Show that  $F$  has a minimum at a certain point  $(x_0, y_0) \in \mathbb{R}^2$ , find it, and calculate the minimum value  $F(x, y)$  for  $(x, y) \in \mathbb{R}^2$ .

**Solution** The system

$$0 = F_x = 2x + y + 1. \quad 0 = F_y = x + 2y + 2,$$

yields that

$$x = 0 \quad y = -1,$$

is the only stationary point. Observe that

$$F_{xx}(x, y) = 2 > 0 \quad F_{yy}(x, y) = 2 > 0, \quad \forall (x, y) \in \mathbb{R}^2,$$

and that  $F_{xy}(x, y) = F_{yx}(x, y) = 1$ . Thus, the determinant of the Hessian is

$$D(x, y) = F_{xx}F_{yy} - F_{xy}^2 = 4 - 1 = 3 > 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

Hence,  $(0, -1)$  is a global minimum point for  $F$  by one of theorems seen in the lectures (Theorem 13.2.1 in the course-book). Moreover

$$\min_{(x, y) \in \mathbb{R}^2} F(x, y) = F(0, -1) = 13.$$

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**GOOD LUCK!**

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*The corrected exams will be handed out on Wednesday, November 10 2015, at 10:30, in the room next to the coffee shop, house 5, and after that in room 204, house 6.*