1. (a) $\lim_{t\to 0} \dfrac{1-e^{2t}}{1-e^t} \overset{?}{=} \dfrac{\lim_{t\to 0}(1-e^{2t})}{\lim_{t\to 0}(1-e^t)} \overset{?}{=} \dfrac{0}{0}$

since $(1-e^{2t})$, $(1-e^t)$ are continuous fns.

✗ Indeterminate form! Use l'Hospital:

$$\ldots = \lim_{t\to 0} \left(\frac{\frac{d}{dt}(1-e^{2t})}{\frac{d}{dt}(1-e^t)} \right.$$

$$= \lim_{t\to 0} \frac{-2e^{2t}}{-e^t} = \frac{-2e^0}{-e^0}$$

All continuous functions

$$= \frac{-2}{-1} = \boxed{2}$$

(b) $\lim_{t\to 0} \dfrac{1-2t}{1-t} = \dfrac{1-0}{1-0} = \boxed{1}$

2. $f(x) = \begin{vmatrix} 1+x & 1+x \\ x & 4 \end{vmatrix} = 4(1+x) - x(1+x) = (4-x)(1+x)$

Completing the square, to more easily find max. value etc.

$$= 4 - 3x - x^2$$

$$= 4 - (x^2+3x)$$

$$= 4 - \left(x^2+3x+\left(\tfrac{3}{2}\right)^2\right) + \left(\tfrac{3}{2}\right)^2$$

$$= 4 + \frac{9}{4} - \left(x+\frac{3}{2}\right)^2$$

So maximum value $f(x_*)$ occurs when $x_* + 3/2 = 0$, i.e. $x_* = -\frac{3}{2}$,

& there, $f(x_*) = 4 + \frac{9}{4} = \frac{25}{4}$.

So we have (a) $\boxed{f(x_*) = \dfrac{25}{4}}$

and (b)
$$
\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} (4-x)(1+x) \\
&\overset{?}{=} \left(\lim_{x\to\infty} (4-x)\right)\left(\lim_{x\to\infty} (1+x)\right) \\
&\overset{?}{=} \text{"}\cancel{\lim}\,(4-\infty)(1+\infty)\text{"} \\
&\overset{?}{=} \text{"}(-\infty)(\infty)\text{"}
\end{aligned}
$$

$$\boxed{\lim_{x\to\infty} f(x) = -\infty}$$

} Not really valid algebra, but following the rules for limits of sums & products.

3. Use Gaussian elimination for both:

(a)
$$\left(\begin{array}{cccc|c} 2 & -2 & 0 & 1 & 4 \\ 1 & 0 & -3 & 1 & 3 \end{array}\right)$$
$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & \frac{1}{2} & 2 \\ 1 & 0 & -3 & 1 & 3 \end{array}\right)$$
$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & -3 & \frac{1}{2} & 1 \end{array}\right)$$
$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & -3 & \frac{1}{2} & 1 \end{array}\right)$$

<u>So</u> original eq'ns are equivalent to:
$$
\begin{aligned}
x \quad -3z + w &= 3 \\
y - 3z + \tfrac{1}{2}w &= 1.
\end{aligned}
$$

So the solutions are

$$\boxed{\begin{aligned} x &= 3 + 3s - t \\ y &= 1 + 3s - t/2 \\ z &= s \quad \text{for any} \\ w &= t \quad s,t \in \mathbb{R}. \end{aligned}}$$

(b)
$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 9 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 9 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 6 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Last equation is now "$0 = -1$",

$\boxed{\text{so original equations are } \underline{\text{inconsistent}} \longrightarrow \text{no solutions!}}$

4. $f(x,y) = x^3 - 3x - (x+y)^2$

So $f_x(x,y) = 3x^2 - 3 - 2(x+y)$

$f_y(x,y) = -2(x+y)$

So stat. points occur when ① $3x^2 - 3 - 2(x+y) = 0$

② $-2(x+y) = 0$.

Substituting ② in ① gives $3x^2 - 3 = 0$

$\iff x^2 - 1 = 0$

$\iff x^2 = 1$

$\iff x = \pm 1$.

Meanwhile, ② is equivalent to

$x + y = 0$

$\iff y = -x$.

So stat. points of f are $(1,-1)$ and $(-1,1)$

Hessian $H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x-2 & -2 \\ -2 & -2 \end{pmatrix}$

so $|H(f)| = (-2)(6x-2) - (-2)(-2)$

$= -2\,(6x - 2 - (-2)) = -2\,(6x) = -12x$.

$$\boxed{\begin{array}{l}\text{So at } (1,-1), \quad |H(f)| = -12 < 0, \\ \quad \text{so this is a } \underline{\text{saddle point}}; \\ \text{at } (-1,1), \quad |H(f)| = 12 > 0, \\ \quad \text{and } f_{xx} = 6(-1) - 2 = -8 < 0 \\ \quad \text{so this is a (strict local) } \underline{\text{maximum}}.\end{array}}$$

(b) The region $\{(x,y) \mid x \geq 0\}$ is not $\underline{\text{bounded}}$,

so the EVT does not apply to it.

5. $$\underbrace{4(u+v)^2 + u - v}_{\text{Call this } F(u,v)} = 14$$

Intersection with V-axis is where $u = 0$,

i.e. $F(0,V) = 14$

$$4V^2 - V = 14$$

$$\cancel{V^2 - \frac{V}{4} - \frac{14}{4} = 0} \qquad \underbrace{4V^2 - V - 14}_{aV^2 + bV + c} = 0$$

By the quadratic formula. $V = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

so
$$V = \frac{1 \pm \sqrt{1-4(4)(-14)}}{2\cdot 4} = \frac{1 \pm \sqrt{1+16\cdot 14}}{8} = \frac{1 \pm \sqrt{225}}{8} = \frac{1 \pm 15}{8}$$

so $V = \frac{16}{8} = 2$ or $V = -\frac{14}{8}$.

so the V-intercepts are at $\boxed{(0,2) \text{ and } (0,-\tfrac{14}{8})}$

(b) Differentiation along a level curve $F(U,V) = c$:

$$\cancel{\frac{dV}{dU} = -\frac{\left(\frac{\partial F}{\partial U}\right)}{\left(\frac{\partial F}{\partial V}\right)}}$$

Since U is locally a function of V, we find $\frac{dU}{dV}$:

$$\frac{dU}{dV} = -\frac{F_V(U,V)}{F_U(U,V)} = -\frac{8(U+V)-1}{8(U+V)+1}$$

At (U_*,V_*):
$$= -\frac{8\left(\frac{1}{32}+7+\frac{3}{32}-7\right)-1}{8\left(\frac{1}{32}+7+\frac{3}{32}-7\right)+1} = -\frac{8\cdot\frac{4}{32}-1}{8\cdot\frac{4}{32}+1} = -\frac{1-1}{1+1} \boxed{=0}.$$

So at (U_*, V_*), $\frac{dU}{dV} = 0$,

so equation of tangent line is

$$U = \frac{dU}{dV}(V - V_*) + U_*$$

$$= 0\,(V - V_*) + U_*$$

$$= U_*$$

$$\boxed{U = \frac{1}{32} + 7}$$

6. (a) $\displaystyle\int_0^a x e^{-x^2}\,dx = \int_{x=0}^{x=a} -\frac{1}{2} e^u\,du$

Substitution: $u = -x^2$, $\frac{du}{dx} = -2x$

$$= \int_{u=0}^{u=-a^2} -\frac{1}{2} e^u\,du$$

$$= \left[-\frac{1}{2}e^u\right]_0^{-a^2} = \frac{1}{2}\left(-e^{-a^2} + e^{-0}\right) = \boxed{\frac{1}{2}\left(1 - e^{-a^2}\right)}$$

(b) $\displaystyle\int_0^\infty x e^{-x^2}\,dx = \lim_{a\to\infty} \int_0^a x e^{-x^2}\,dx$

$$= \lim_{a\to\infty} \frac{1}{2}\left(1 - e^{-a^2}\right) = \frac{1}{2}\left(1 - \lim_{a\to\infty} e^{-a^2}\right)$$

by part (a)

$$= \frac{1}{2}(1 - 0) \boxed{= \frac{1}{2}}$$

Int. by parts:

$u = x \qquad \frac{dv}{dx} = 2^x \ (= e^{\ln 2 x})$

$\frac{du}{dx} = 1 \qquad v = \int 2^x dx = \frac{2^x}{\ln 2} \ (+c)$

(c) $\int x 2^x dx = \frac{x 2^x}{\ln 2} - \int \frac{2^x}{\ln 2} dx$

$= \frac{x 2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + C$

$$\boxed{= \frac{2^x}{\ln 2}\left(x - \frac{1}{\ln 2}\right) + C}$$