Mathematics for Economic and Statistical Analysis
Solutions Exam 2017/10/30

1) (a) Substituting x by 0 we get an indeterminate of the type $\frac{0}{0}$:

$$\lim_{x\to 0} \frac{x\log(1+x)}{3(1+x-e^x)} = \frac{0}{0}.$$

Then we observe that both functions

$$f(x) = x\log(1+x) \quad \text{and} \quad g(x) = 3(1+x-e^x)$$

are differentiable in the open interval $(-1,1)$, which contains 0. Hence we can apply l'Hôpital's rule to get that

$$\lim_{x\to 0} \frac{x\log(1+x)}{3(1+x-e^x)} = \lim_{x\to 0} \frac{\log(1+x)+\frac{x}{1+x}}{3(1-e^x)},$$

which produces again an indeterminate of the type $\frac{0}{0}$. Arguing as before, we can apply l'Hôpital's rule again to get that

$$\lim_{x\to 0} \frac{x\log(1+x)}{3(1+x-e^x)} = \lim_{x\to 0} \frac{\log(1+x)+\frac{x}{1+x}}{3(1-e^x)} = \lim_{x\to 0} \frac{\frac{1}{1+x}+\frac{1}{(1+x)^2}}{-3e^x} = -\frac{2}{3}.$$

(b) We multiply both the numerator and the denominator by the reciprocal of the highest power, that is, $1/x^3$. So we get that

$$\begin{aligned}
\lim_{x\to\infty} \frac{50x^2}{x^3-x+1} &= \lim_{x\to\infty} \frac{\frac{1}{x^3}(50x^2)}{\frac{1}{x^3}(x^3-x+1)} \\
&= \lim_{x\to\infty} \frac{50\frac{x^2}{x^3}}{\frac{x^3}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{50}{x}}{1-\frac{1}{x^2}+\frac{1}{x^3}} \\
&= \frac{0}{1-0+0} \\
&= 0.
\end{aligned}$$

2) We must maximize and minimize the function $P(T) = e^T(T^2-4T+1)$ in the interval $[0,4]$.

First of all we find the stationary points, that is, the solutions to the equation $P'(T) = 0$. The derivative of P is

$$P'(T) = e^T(T^2 - 4T + 1) + e^T(2T - 4) = e^T(T^2 - 2T - 3),$$

and the solutions to the equation $P'(T) = 0$ are the solutions to the second degree equation $T^2 - 2T - 3 = 0$, since e^T cannot be 0.

We apply the quadratic formula (the p, q formula) to solve $T^2 - 2T - 3 = 0$, obtaining

$$T = \frac{2}{2} \pm \sqrt{\frac{4}{4} + 3} = 1 \pm 2 = \begin{cases} 3 \\ -1 \end{cases}.$$

We observe that -1 does not belong to the interval $[0, 4]$ and hence $T = 3$ is the unique stationary point.

Finally, we compute the value of $P(T)$ at the extremes of the interval ($T = 0$, $T = 4$) and at the stationary point ($T = 3$):

$$P(0) = 1, \quad P(4) = e^4, \quad P(3) = -2e^3.$$

Comparing the values, we conclude that e^4 is the highest value that P can attain when $0 \leq T \leq 4$, while $-2e^3$ is the lowest value that P can attain when $0 \leq T \leq 4$.

3) The stationary points of g are all those points (x, y) satisfying the nonlinear system of equations

$$\begin{cases} \frac{\partial g}{\partial x}(x, y) = 3x^2 + 3y = 0, \\ \\ \frac{\partial g}{\partial y}(x, y) = 3y^2 + 3x = 0. \end{cases}$$

From the first equation we deduce that $y = -x^2$. Plugging this relation in the second equation, we get

$$3x^4 + 3x = 3x(x^3 + 1) = 0,$$

whose solutions are $x_1 = 0$ and $x_2 = \sqrt[3]{-1} = -1$. Then we observe that

$$\begin{cases} x_1 = 0 \Rightarrow y_1 = -x_1^2 = 0, \\ x_2 = -1 \Rightarrow y_2 = -x_2^2 = -1. \end{cases}$$

We get then two stationary points: $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (-1, -1)$. To check if those points are local maximum, local minimum or saddle points we must apply the second-derivative test for local extrema. The partial derivatives of order two are

$$\frac{\partial^2 g}{\partial x^2}(x,y) = 6x, \quad \frac{\partial^2 g}{\partial y^2}(x,y) = 6y \quad \text{and} \quad \frac{\partial^2 g}{\partial x \partial y}(x,y) = \frac{\partial^2 g}{\partial y \partial x}(x,y) = 3.$$

For the point $(x_1, y_1) = (0,0)$ we observe that

$$\left(\frac{\partial^2 g}{\partial x^2}(0,0)\right) \cdot \left(\frac{\partial^2 g}{\partial y^2}(0,0)\right) - \left(\frac{\partial^2 g}{\partial x \partial y}(0,0)\right)^2 = 0 - 3^2 < 0,$$

from where we deduce that $(x_1, y_1) = (0,0)$ is a saddle point. On the other hand, if we pick the point $(x_2, y_2) = (-1,-1)$ we observe that

$$\frac{\partial^2 g}{\partial x^2}(-1,-1) = -6 < 0$$

and

$$\left(\frac{\partial^2 g}{\partial x^2}(-1,-1)\right) \cdot \left(\frac{\partial^2 g}{\partial y^2}(-1,-1)\right) - \left(\frac{\partial^2 g}{\partial x \partial y}(-1,-1)\right)^2 = 36 - 9 > 0,$$

from where we deduce that $(x_2, y_2) = (-1,-1)$ is a local maximum point.

4) We compute the products appearing in both sides of the equation:

$$\begin{pmatrix} 3a + 6b + 2 \\ -a + 2b - 6 \\ 4a + b + 4 \end{pmatrix} = \begin{pmatrix} b + 2c + 3 \\ -b - 5c - 6 \\ b + 5c + 3 \end{pmatrix}.$$

Ordering the terms we get the following linear system of equations:

$$\begin{aligned} 3a + 5b - 2c &= 1, \\ -a + 3b + 5c &= 0, \\ 4a - 2b - 5c &= -1, \end{aligned}$$

which can be written in matrix form as

$$\left(\begin{array}{ccc|c} 3 & 5 & -2 & 1 \\ -1 & 3 & 5 & 0 \\ 4 & -2 & -5 & -1 \end{array}\right).$$

We find the solution to this system by using Gaussian elimination. First of all, we change sign in the second row and we interchange it with the first one, obtaining

$$\left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 3 & 5 & -2 & 1 \\ 4 & -2 & -5 & -1 \end{array}\right).$$

Next we subtract to the second row the first one multiplied by 3, while we subtract to the third row the first one multiplied by 4, obtaining

$$\left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 14 & 13 & 1 \\ 0 & 10 & 15 & -1 \end{array}\right).$$

Now we divide the second row by 14, obtaining

$$\left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & \frac{13}{14} & \frac{1}{14} \\ 0 & 10 & 15 & -1 \end{array}\right).$$

Finally, we substract to the third row the second one multiplied by 10, obtaining

$$\left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & \frac{13}{14} & \frac{1}{14} \\ 0 & 0 & \frac{40}{7} & \frac{-12}{7} \end{array}\right).$$

Th simplified system of equations is

$$\begin{aligned} a - 3b - 5c &= 0, \\ b + \frac{13}{14}c &= \frac{1}{14}, \\ \frac{40}{7}c &= \frac{-12}{7}. \end{aligned}$$

From the last equation we get that $c = -3/10$.

Replacing the value of c in the second equation we get that $b = 7/20$.

Replacing the value of b and c in the first equation we get that $a = -9/20$.

5) (a) We change variables $x = y^2 - 1$ (and then $\mathrm{d}x = 2y\mathrm{d}y$) and we observe that $x = -1$ when $y = 0$ and $x = 0$ when $y = 1$. Hence we have the following:

$$\begin{aligned} \int_0^1 ye^{y^2-1}\mathrm{d}y &= \int_{-1}^0 ye^x \frac{\mathrm{d}x}{2y} = \frac{1}{2}\int_{-1}^0 e^x \mathrm{d}x = \frac{1}{2}\left[e^x\right]_{x=-1}^{x=0} = \frac{1}{2}(1 - e^{-1}) \\ &= \frac{1}{2} - \frac{1}{2e}. \end{aligned}$$

(b) We use the integration by parts formula:

$$\int uv' = uv - \int u'v.$$

We pick $u = 3\log(t)$ and $v' = 1/\sqrt{t}$, from where $u' = 3/t$ and $v = 2\sqrt{t}$. Then we have that

$$\begin{aligned}\int \frac{3\log(t)}{\sqrt{t}}\mathrm{d}t &= 3\log(t)2\sqrt{t} - \int \frac{3}{t}2\sqrt{t}\mathrm{d}t \\ &= 6\sqrt{t}\log(t) - 6\int \frac{\sqrt{t}}{t}\mathrm{d}t \\ &= 6\sqrt{t}\log(t) - 6\cdot 2\sqrt{t} \\ &= 6\sqrt{t}(\log(t) - 2).\end{aligned}$$

6) (a) We differentiate the equation with respect to x, obtaining

$$2xy^3 + 3x^2y^2y' + 10x^4y + 2x^5y' = 3y' + 12.$$

We evaluate the expression at the point $(x, y) = (-1, 1)$ (so we make $x = -1$ and $y = 1$) to get

$$\begin{aligned}-2 + 3y' + 10 - 2y' &= 3y' + 12 \\ 3y' - 2y' - 3y' &= 12 + 2 - 10 \\ y' &= -2.\end{aligned}$$

(b) If $y = 0$ then the equation becomes

$$0 = 12x + 8,$$

from where $x = -2/3$.