MH100S Exam 2018-10-03 Solutions

1. (a) $$\lim_{x\to 2} \frac{3^x - \cancel{2}x^2}{x^2-3} = \frac{3^2-2^2}{2^2-3} = \frac{5}{1} = \boxed{5}$$

(b) $$\lim_{x\to 2} \frac{2^x - x^2}{x^2-4} \overset{(?)}{=} \frac{2^2-2^2}{2^2-4} \text{ "}= \frac{0}{0}\text{"} \quad \times$$

Indeterminate form!

So l'Hospital applies:

$$\dots = \lim_{x\to 2} \frac{\frac{d}{dx}(2^x - x^2)}{\frac{d}{dx}(x^2-4)}$$

$$= \lim_{x\to 2} \frac{(\ln 2)2^x - 2x}{2x}$$

$$= \frac{(\ln 2)2^2 - 2\cdot 2}{2\cdot 2} \boxed{= \ln 2 - 1}$$

(c) $$\lim_{x\to 2^+} \frac{3^x - x^2}{x^2-4} \overset{(?)}{=} \frac{3^2-\cancel{2}^2}{2^2-4} = \text{"}\frac{5}{0}\text{"}$$: not finite, so check more carefully, for one-sided limit.

$$\lim_{x\to 2^+} (3^x - x^2) = 3^2 - 2^2 = 5$$

$$\lim_{x\to 2^+} (x^2-4) = 0^+ \quad (\text{since } x^2-4>0, \text{ for } x>2)$$

so $$\lim_{x\to 2^+} \frac{3^x-x^2}{x^2-4} = \text{"}\left(\frac{5}{0^+}\right)\text{"} = \boxed{\infty}.$$

2. $f(x) = \frac{x^2+2}{x} = x + \frac{2}{x}$, defined on $\mathbb{R}\setminus\{0\}$

$f'(x) = 1 - \frac{2}{x^2}$

(a) Max/min values are either at

- critical points of f
- discontinuities of f
- endpoints of the interval, $[\frac{1}{2}, 3]$.

Crit pts:
$$f'(x) = 0$$
$$1 - \frac{2}{x^2} = 0$$
$$1 = \frac{2}{x^2}$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

$-\sqrt{2}$ is outside the given interval, but $\sqrt{2}$ is inside it (we know $1 < \sqrt{2} < 2$ since $1^2 < 2 < 2^2$).

Discontinuities: Only discont. of f is 0, which is not in the interval.

So: possible max/min of f on $[\frac{1}{2}, 3]$ are $x = \frac{1}{2}, \sqrt{2}, 3$:

x	$\frac{1}{2}$	$\sqrt{2}$	3
$f(x)$	$9/2$	$2\sqrt{2}$	$3\frac{2}{3}$

2 (cont'd) $\quad 2\sqrt{2} < 3$ $\quad$ (since $\sqrt{2} < \frac{3}{2}$,

as $2 < \left(\frac{3}{2}\right)^2$)

so $\quad 2\sqrt{2} < 3\frac{2}{3} < \frac{9}{2}$,

$$\boxed{\begin{array}{l} \text{so } \underline{\text{minimum}} \text{ of } f \text{ on } [\tfrac{1}{2}, 3] \text{ is } f(\sqrt{2}) = 2\sqrt{2}, \\ \& \ \underline{\text{max}} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad f(\tfrac{1}{2}) = \tfrac{9}{2}. \end{array}}$$

(b) As above, $f'(x) = 1 - \frac{2}{x^2}$

$$1 - \frac{2}{x^2} \geq 0 \iff 1 \geq \frac{2}{x^2}$$

$$\iff x^2 \geq 2$$

So $f'(x) > 0$ for $x > \sqrt{2}$

and $f'(x) < 0$ for $-\sqrt{2} < x < \sqrt{2}$

$$\boxed{\begin{array}{l} \text{So within the given interval,} \\ \quad f \text{ is (strictly) increasing on } [\sqrt{2}, 3) \\ \quad \text{and (strictly) decreasing on } [\tfrac{1}{2}, \sqrt{2}] \end{array}}$$

3 For each part, use Gaussian elimination:

(a)
$$\begin{array}{c} \\ R1 \\ R2 \\ R3 \end{array}
\begin{array}{c}
\begin{array}{ccc} a & b & c \end{array} \\
\left[\begin{array}{ccc|c} 6 & 0 & 2 & 6 \\ -3 & 0 & -2 & 0 \\ 1 & -1 & -1 & 2 \end{array}\right]
\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & -1 & 3 \\ 0 & -1 & -\frac{4}{3} & 1 \end{array}\right]
\quad \begin{array}{l} \text{Multiply R1 by } \frac{1}{6} \\ \text{Add } 3 \times \text{R1 to R2} \\ \text{Add } (-1) \times \text{R1 to R3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & -1 & -\frac{4}{3} & 1 \\ 0 & 0 & -1 & 3 \end{array}\right]
\quad \text{Swap R2, R3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{4}{3} & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]
\quad \text{Multiply R2, R3 by } (-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right]
\quad \begin{array}{l} \text{Add } (-\frac{1}{3}) \times \text{R3 to R1} \\ \text{Add } (-\frac{4}{3}) \times \text{R3 to R1} \end{array}$$

$$\boxed{\text{General solution: } a = 2, \quad b = 3, \quad c = -3.}$$

(Check in original equations: $6 \cdot 2 + 2 \cdot (-3) = 12 - 6 = 6$ ✓

$-3 \cdot 2 - 2 \cdot (-3) = -6 + 6 = 0$ ✓

$2 - 3 - (-3) = 2$ ✓)

3 (b)

$$\begin{array}{c|c} & \begin{matrix} x & y & z & w & \end{matrix} \\ \begin{matrix} R1 \\ R2 \\ R3 \end{matrix} & \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -3 \\ 0 & -2 & 0 & 2 & 0 \\ -2 & 1 & -4 & -1 & 6 \end{array}\right] \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -3 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \quad \text{Add } 2 \times R1 \text{ to } R3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \quad \text{Multiply } R2 \text{ by } -\tfrac{1}{2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Subtract } R2 \text{ from } R3$$

$$x + 2z = -3$$
$$y - w = 0$$
$$0 = 0 \quad \text{(redundant)}$$

So the general solution is

$$x = -3 - 2z$$
$$y = w$$

for any choice of $z, w \in \mathbb{R}$.

4 Second-order Taylor expansion of $\sqrt{1+2x}$:

$$f(x) = \sqrt{1+2x} = (1+2x)^{1/2}$$

$$f'(x) = 2 \cdot \tfrac{1}{2}(1+2x)^{-1/2} = \frac{1}{\sqrt{(1+2x)}}$$

$$f''(x) = 2 \cdot \left(-\tfrac{1}{2}\right)(1+2x)^{-3/2} = \frac{-1}{(1+2x)^{3/2}}$$

$$f^{(3)}(x) = 2 \cdot (-1) \cdot \left(-\tfrac{3}{2}\right)(1+2x)^{-5/2} = \frac{3}{(1+2x)^{5/2}}.$$

So $$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$
$$= 1 + x - \tfrac{1}{2}x^2.$$

By Lagrange remainder form,

for any $x \geq 0$, $R_3(x) = \frac{f^{(3)}(c)x^3}{3!}$ for some $c \in (0, x)$

$$= \frac{1}{2(1+2c)^{5/2}}x^3$$

Since $c \geq 0$, $(1+2c) \geq 1$

so $(1+2c)^{5/2} \geq 1$

so $0 \leq R_3(x) \leq \frac{1}{2}x^3$ for any $x \geq 0$,

i.e. $0 \leq \sqrt{1+2x} - (1 + x - \frac{1}{2}x^2) \leq \frac{1}{2}x^3$

i.e. $1 + x - \frac{1}{2}x^2 \leq \sqrt{1+2x} \leq 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$,

as required.

5 (a) $\int t^2 \ln(2t)\, dt$

By parts:
$u = \ln(2t) \quad \frac{dv}{dt} = t^2$
$\frac{du}{dt} = \frac{1}{t} \quad v = \frac{t^3}{3}$

$$= \frac{t^3 \ln(2t)}{3} - \int \left(\frac{t^3}{3}\right)\left(\frac{1}{t}\right) dt$$

$$= \frac{1}{3}\left(t^3 \ln 2t - \int t^2\, dt\right)$$

$$= \frac{1}{3}\left(t^3 \ln 2t - \frac{t^3}{3}\right) + C$$

$$= \frac{t^3}{3}\left(\ln 2t - \frac{1}{3}\right) + C$$

(b) $\displaystyle\int_1^e \frac{\ln y}{y\left((\ln y)^2 + 2\right)}\, dy$

Substitution:
$u = \ln y$
$\frac{du}{dy} = \frac{1}{y}$
$y = e \Leftrightarrow u = 1$
$y = 1 \Leftrightarrow u = 0$

$$= \int_{y=1}^{y=e} \frac{u}{u^2+2} \frac{du}{dy}\, dy$$

$$= \int_{u=0}^{u=1} \frac{u}{u^2+2}\, du$$

$v = u^2 + 2$
$\frac{dv}{du} = 2u$
$u = 1 \Leftrightarrow v = 3$
$u = 0 \Leftrightarrow v = 2$

$$= \int_{u=0}^{u=1} \left(\frac{1}{v}\right) \frac{1}{2} \frac{dv}{du}\, du$$

$$= \frac{1}{2}\int_{v=2}^{v=3} \frac{1}{\cancel{2}v}\, dv = \frac{1}{2}\Big[\ln v\Big]_2^3 = \frac{1}{2}(\ln 3 - \ln 2)$$

$$= \frac{1}{2} \ln \frac{3}{2}.$$

6. $$g(x,y) = x^2 - 2x + y^2 + C(xy - y).$$
$$g_x(x,y) = 2x - 2 + Cy$$
$$g_y(x,y) = 2y + Cx - C.$$

(a) $g_x(x,y) = 0 \iff 2x - 2 + Cy = 0$
$$\iff x = \frac{2 - Cy}{2} = 1 - \frac{C}{2}y$$

$g_y(x,y) = 0 \iff 2y + Cx - C = 0$
$$\iff y = \frac{C - Cx}{2} = \frac{C}{2} - \frac{C}{2}x.$$

So crit. pt. occurs just when

① $x = 1 - \frac{C}{2}y$

and ② $y = \frac{C}{2} - \frac{C}{2}x$.

Substituting ① in ②,
$$y = \frac{C}{2} - \frac{C}{2}\left(1 - \frac{C}{2}y\right)$$
$$\iff y = \frac{C}{2} - \frac{C}{2} + \frac{C^2}{4}y$$
$$\iff \left(1 - \frac{C^2}{4}\right)y = 0$$
$$\iff y = 0 \quad \left(\text{since } 1 - \frac{C^2}{4} \neq 0, \text{ as } C \neq \pm 2\right)$$

6 (a) cont'd. So ① implies

$$x = 1 - \frac{C}{2}y = 1 - \frac{C}{2}0$$

$$x = 1$$

$$\boxed{\text{So for } C \neq \pm 2, \ (1,0) \text{ is the unique critical point.}}$$

(b) Kind of crit. pt. depends on

signs of $\underbrace{\begin{vmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{vmatrix}}_{\text{"}|H_g(x,y)|\text{"}}$ & g_{xx}.

$$\begin{vmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{vmatrix} = \begin{vmatrix} 2 & C \\ C & 2 \end{vmatrix}$$

$$= 4 - C^2.$$

$$4 - C^2 > 0$$

$$\Leftrightarrow \quad C^2 < 4$$

$$\Leftrightarrow \quad |C| < 2. \qquad \& \ g_{xx} = 2 > 0 \text{ for any } C.$$

$$\boxed{\begin{array}{l} \underline{\text{So}}\text{: for } C < -2 \text{ and } C > 2, \\ \text{the crit pt. is a } \underline{\text{saddle pt}}\text{, as } |H_g(x,y)| < 0\,; \\ \text{for } -2 < C < 2, \text{ the crit pt is a } \underline{\text{minimum}}, \\ \text{as } |H_g(x,y)| > 0 \text{ and } g_{xx}(x,y) > 0. \end{array}}$$