1 Solution exam MM1005, HT18, 7.5 ECTS, 7 November 2018

1.

(a) We manipulate the expression in order to obtain a common denominator

$$\lim_{x\to\infty} \frac{x^2}{2-x} + x = \lim_{x\to\infty} \frac{x^2 + x(2-x)}{2-x},$$
$$= \lim_{x\to\infty} \frac{2x}{2-x},$$

at this point divide both numerator and denominator by x and show that the limit exists finite

$$= \lim_{x\to\infty} \frac{2}{\frac{2}{x} - 1} = -2.$$

(b) By substituting x with 0 we obtain the indeterminate form of type $0/0$. Because both the numerator and denominator are differentiable functions then l'Hôpital's rule can be applied

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{x\ln(2+x)} = \lim_{x\to 0} \frac{2xe^{x^2}}{\ln(2+x) + \frac{x}{2+x}},$$
$$= \frac{0}{\ln(2) + \frac{0}{2}} = 0.$$

2.

(a) By setting $x = 1$ we obtain the equation $y^3 + y = 10$ which has only one solution $y = 2$.

(b) We compute the derivative of the implicitly defined function by differentiating with respect to x both side of the equation while treating y like a function in x

$$\frac{d}{dx}(y^3 + x^2y - 10) = 3y^2y' + 2xy + x^2y' = 0,$$

hence substitute $x = 1$ and $y = 2$ to obtain

$$12y' + 4 + y' = 0$$

whose solution gives $y'(1) = -\frac{4}{13}$. The equation of the tangent line to the graph of a function f at the point $a = 1$ is given by $y - f(1) = f'(1)(x-1)$ hence in this case $y - 2 = -\frac{4}{13}(x-1)$, i.e.

$$y = -\frac{4}{13}x + \frac{30}{13}.$$

3.

(a) The integrand is continuous on the close interval $[1,2]$ then it is integrable. Notice that $1/x$ is the derivative of $1+\ln x$, hence set $g(x)=1+\ln x$ and perform a change of variable

$$\begin{aligned}\int_1^2 \frac{\sqrt{1+\ln x}}{x}dx &= \int_1^2 \sqrt{g(x)}g'(x)dx,\\ &= \int_{g(1)=1}^{g(2)=1+\ln 2} \sqrt{u}du,\\ &= \frac{2}{3}u^{\frac{3}{2}}\Big|_1^{1+\ln 2} = \frac{2}{3}\left((1+\ln 2)^{\frac{3}{2}}-1\right).\end{aligned}$$

(b) The integrand is continuous on the interval $(0,1]$, but it is not defined at the point $x=0$. Hence the integral is improper and we compute the following limit and check if it is finite

$$\lim_{a\to 0^+}\int_a^1 x^2(\ln x)dx.$$

The computation of the integral requires to integrate by parts

$$\begin{aligned}\int_a^1 \underbrace{x^2}_{f'}\underbrace{(\ln x)}_{g}dx &= \underbrace{\frac{x^3}{3}}_{f}\underbrace{\ln x}_{g}\Bigg|_a^1 - \int_a^1 \underbrace{\frac{x^3}{3}}_{f}\underbrace{\frac{1}{x}}_{g'}dx,\\ &= 0-\frac{a^3}{3}\ln a - \frac{x^3}{9}\Big|_a^1 = -\frac{a^3}{3}\ln a - \frac{1}{9}+\frac{a^3}{9};\end{aligned}$$

then the limit of the integral can be computed using l'Hôpital's rule as follows

$$\begin{aligned}\lim_{a\to 0^+}\int_a^1 x^2(\ln x)dx &= \lim_{a\to 0^+} -\frac{a^3}{3}\ln a - \frac{1}{9}+\frac{a^3}{9},\\ &= -\frac{1}{9} - \lim_{a\to 0^+}\frac{\ln a}{3a^{-3}}, \qquad (1)\\ &= -\frac{1}{9} - \lim_{a\to 0^+}\frac{a^{-1}}{-9a^{-4}} = -\frac{1}{9}.\end{aligned}$$

Therefore we conclude that the improper integral converges to $-1/9$.

4.

(a) A square matrix is invertible if and only if its determinant is different from zero, therefore we compute the determinant

$$\det \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & c \\ 1 & c & 1 \end{pmatrix} = 1 \cdot (4 - c^2) - 2 \cdot (2 - c) = c(2 - c).$$

The solutions to the equation $c(2 - c) = 0$ are $c = 0, 2$ hence the matrix is invertible when $c \neq 0, 2$.

(b) From the previous point we see that the determinant of the matrix is zero, hence the system has either zero or infinitely many solutions. By subtracting twice the third row from the second row and later subtracting once the first row from the third we obtain the following new system

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \tag{2}$$

which has infinitely many solutions of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - 2t \\ t \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}. \tag{3}$$

5.

(a) $g(x) = \sqrt[3]{1 + x}$ then $g'(x) = \frac{1}{3}(1 + x)^{-2/3}$ and $g''(x) = -\frac{2}{9}(1 + x)^{-5/3}$, hence

$$\begin{aligned} T_2(x) &= 1 + \frac{1}{1!}\frac{1}{3}x - \frac{1}{2!}\frac{2}{9}x^2 \\ &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 \end{aligned} \tag{4}$$

(b) The remainder is given by the following expression

$$\begin{aligned} R_3(x) &= g(x) - T_2(x), \\ &= \frac{g'''(c)}{3!}x^3, \\ &= \frac{\frac{10}{27}(1 + c)^{-8/3}}{6}x^3, \\ &= \frac{5}{81}(1 + c)^{-8/3}x^3 \end{aligned}$$

where $c \in [0, x]$ depends on x. Since $|(1 + c)^{-8/3}| \leq 1$ for $c \in [0, x]$ then

$$|R_3(x)| = \left| \frac{5}{81}(1 + c)^{-8/3}x^3 \right| \leq \frac{5}{81}x^3.$$

(c) We rewrite the expression by making use of the function $g(x)$ in such a way that the value of x is smaller than 1 in order to have the error term as small as possible. By following the hint

$$\sqrt[3]{1002} = \sqrt[3]{1000\left(1+\frac{2}{1000}\right)} = 10\sqrt[3]{1+\frac{2}{1000}} = 10g\left(\frac{2}{1000}\right),$$

then apply (4) and observe that

$$\begin{aligned} 10g\left(\frac{2}{1000}\right) &= 10\left(T_2\left(\frac{2}{1000}\right) + R_3\left(\frac{2}{1000}\right)\right), \\ &= 10\left(1+\frac{1}{3}\frac{2}{1000} - \frac{1}{9}\left(\frac{2}{1000}\right)^2 + R_3\left(\frac{2}{1000}\right)\right), \\ &= 10\left(1+0.000\overline{6} - 0.000000\overline{4} + R_3\left(\frac{2}{1000}\right)\right), \\ &= 10.00666\overline{2} + 10R_3\left(\frac{2}{1000}\right). \end{aligned}$$

Hence

$$\begin{aligned} \left|\sqrt[3]{1002} - 10.00666\overline{2}\right| = \left|10R_3\left(\frac{2}{1000}\right)\right| &\le \frac{5}{81}\left(\frac{2}{1000}\right)^3, \\ &\le \frac{1}{10}\cdot 10\left(\frac{1}{1000}\right)^3 = 10^{-9}, \end{aligned}$$

where in the last step we have used $\frac{5}{81} < \frac{1}{10}$ and $2^3 = 8 < 10$. Therefore

$$\begin{aligned} \left|\sqrt[3]{1002} - 10.0066622\right| &= \left|\sqrt[3]{1002} - 10.00666\overline{2} \quad + 0.\overline{2}\cdot 10^{-7}\right| \\ &\le \left|\sqrt[3]{1002} - 10.00666\overline{2}\right| \quad + \left|0.\overline{2}\cdot 10^{-7}\right| \\ &\le 10^{-9} \quad + 0.3\cdot 10^{-7} \\ &\le \quad + 0.4\cdot 10^{-7} < 0.5\cdot 10^{-7}, \end{aligned}$$

which shows that the 7th decimal digit is still correct after truncation. Said in other words: the last four lines of inequalities show that since the approximation($10.00666\overline{2}$) is distant 10^{-9} from the true value $\sqrt[3]{1002}$ and the truncation of the approximation to the 7th decimal digit (10.0066622) is distant $0.\overline{2}\cdot 10^{-7}$ from the approximation, still the sum of the two errors together is lower than half of the unit of the last digit, which means that 10.0066622 is a better approximation of $\sqrt[3]{1002}$ than 10.0066621 and 10.0066623, the two other possible candidate for the approximation of $\sqrt[3]{1002}$ up to the 7th digit.

On a scientific calculator one can check $\sqrt[3]{1002} \approx 10.00666222715392\ldots$.

6.

The set on which the problem asks for maximum/minimum values of C has the shape of a rectangle triangle T delimited by the lines $s = 0$, $t = 0$ and $s+t = 2$ (with the boundary included), which is closed and bounded, and since the function C is continuous then it admits a maximum and a minimum point on T by the extreme value theorem.

(I) We start by looking for extreme points inside the triangle, hence we find the stationary points. Let $C = C(s,t)$, then the system

$$\begin{cases} C_1' &= t(2s+t-1) = 0, \\ C_2' &= s(s+2t-1) = 0, \end{cases}$$

has the following four solutions: $(s,t) = (0,0),(0,1),(1,0),(1/3,1/3)$ of which only the last one is lies in the interior of T.

(II) We continue by investigating the largest and smallest values of C on the border of T. Along the sides $s = 0$ and $t = 0$ the function is $C \equiv 0$, while along the hypotenuse ($s+t = 2$) we find $C = s(2-s) = t(2-t)$, hence $0 \leq C \leq 1$, where the max is attained at the point $(s,t) = (1,1)$; in order to see this, set the first derivative equal to zero $\frac{d}{ds}s(2-s) = 2-2s = 0$, (the same argument holds also with t and s exchanged).

(III) Finally we compare the extreme values of C on the boundary $C(0,0) = 0, C(1,1) = 1$ with the values of C at the stationary points inside T, hence $C(1/3,1/3) = -1/27$.

Therefore we conclude that the function $C(s,t)$ over the points (s,t) such that $s \geq 0, t \geq 0$ and $s+t \leq 2$ is bounded between the minimum value of $-1/27$ and maximum value 1 reached at the points $(1/3,1/3)$ and $(1,1)$ respectively.