STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

TENTAMEN
Statistical inference theory
Thursday, 17 March 2011

Exam for the course
Statistical inference theory

Thursday, 17 March 2011 9 - 14

Examiner: Anders Björkström, tel. 16 45 54.
Return of exams: Friday, 25 March, 12.00. Room 32, house 5. If you want to know your outcome beforehand, mail a request to bjorks@math.su.se.
Solutions will be available at www.math.su.se after 2 pm today.
Permitted aids: Calculator.
Requirements: Argumentations must be clear and easy to follow. Limits for the various grades are (includng points from the hand-ins):

A	B	C	D	E
58	52	45	38	33

Uppgift 1

Suppose we have a sample $\mathbf{X} = (X_1, \ldots, X_n)$ from a famiy of distributions with an unknown parameter θ, and suppose that $T = t(\mathbf{X})$ is sufficient for θ. If U is an unbiased estimator for θ, then the statistic V, defined as $V = \mathrm{E}[U|T]$ is also unbiased for θ, and can not have larger variance than U.

a) What is the name of this theorem? (2 p)

b) Prove it. (4 p)

c) Further, the theorem states that if the variance of U is not larger than the variance of V, then U is essentially a function of T. By "essentially" we mean that there is a function g such that the event $U = g(T)$ has probability 1. Prove this! (4 p)

Uppgift 2

a) Define the concept score function and prove that its expected value is zero (under certain regularity conditions). (3 p)

b) An important theorem states that MLestimates often have Gaussian distributions, if based on large samples. In elementary courses in statistics, we learn about the Central Limit Theorem (CLT), which states that the Gaussian distribution arises when a random variable is defined as the sum of many random terms. Describe in broad terms how one comes from CLT to the conclusion that ML estimates are Gaussian (You do not need to prove all details). (7 p)

Uppgift 3

a) Given is a sample $\mathbf{X} = (X_1, \ldots, X_n)$ of a random variable that is N(μ, σ^2), where σ^2 is known. If we regard μ as a random variable, what family of distributions is then a conjugated family for it? (No motivation required) (2 p)

b) Give the motivation for your answer in (a). (5 p)

c) The weight of adult males of a certain species has a mean of 7 kg and a standard deviation of 1 kg. We catch an individual at random, give him the name Donald, and determine his weight using a rather worn-out instrument, that yields Gaussian measurement errors with zero mean and standard deviation 0.3 kg. The instrument shows 6.7 kg. What is the Bayes estimate of Donald's weight, if we assume "absolute error loss", that is, the loss is equal to the absolute size of the error? (3 p)

Uppgift 4

We want to estimate the probability θ for a certain type of experiment to be successful. The a priori distribution for θ is Beta with expected value 0.5 and variance 0.2. The experiment is repeated until we have seven successes. Let Y denote the number of trials necessary (including the last one) . Find the Bayes estimate, assuming quadratic loss, as a function of the outcome y. (10 p)

Hint: A continuous random variable X is said to have a Beta distribution with parameters r and s if X takes values between 0 and 1 and has the density function $f_X(x) = B\, x^{r-1}(1-x)^{s-1}$, where B is a number not involving x. It can be shown that the expected value is $r/(r+s)$ and the variance $rs/((r+s+1)(r+s)^2)$.

Uppgift 5

a) Let $X_1, \ldots, X_n$ be a sample of an exponential distribution with expected value μ. Suppose n is large. Use the Central Limit Theorem to form a two-sided confidence interval for μ with confidence level 95 %. (4 p)

b) How large must n be in order that the 95 % confidence interval has an expected length no more than 0.2μ? (6 p)

Hint: The upper 97.5 % quantile for a standard normal variable is 1.96. (Since no tables are available, I should mention this)

Uppgift 6

We have made one observation ($i.e$, $n = 1$) of a random variable X with probability density function

$$f(x) = C(\theta)(1 - x^2)^\theta, \; -1 \leq x \leq 1$$

where θ is an unknown parameter about which we only know that $\theta > -1$. The function $C(\theta)$ does not depend on x, it only serves the purpose to accomplish $\int f(x)dx = 1$.

a) Derive a uniformly most powerful test at the level $\alpha = 0.05$ of $H_0: \; \theta = 0$ versus the composite alternative hypothesis $H_A: \; \theta > 0$. (5 p)

b) Determine the power for this test when $\theta = 1$. (5 p)

Good luck!