1. Basic Definitions

Let $\mathbb{C}^n$ denote the set of all $n$-tuples of complex numbers, i.e. the set of all elements $z = (z_1, z_2, \ldots, z_n)$ where $z_k \in \mathbb{C}$. $\mathbb{C}^n$ is a (complex) vector space in a natural way, and it also carries a natural scalar product

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j. \quad (1)$$

From this scalar product we also obtain a norm $\|z\| = \langle z, z \rangle^{1/2}$ and a distance function $d(z, w) = \|z - w\|$. Just like calculus for functions $f : \mathbb{R} \to \mathbb{R}$ extends in a natural way to calculus for functions $f : \mathbb{R}^n \to \mathbb{R}$, we can try to extend the theory of analytic functions $f : \mathbb{C} \to \mathbb{C}$ to functions $f : \mathbb{C}^n \to \mathbb{C}$.

Definition 1. A function $f : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}^n$ is open, is called analytic if $f \in C^1(\Omega)$ and $f$ is analytic in each variable $z_k$ separately. We denote the class of functions which are analytic in $\Omega$ by $A(\Omega)$.

A remarkable theorem by Hartog (see Hörmander [1]), states that the assumption that $f \in C^1(\Omega)$ is superfluous. Assuming analyticity in each variable separately turns out to imply continuity and in fact infinite differentiability. However, to avoid depending on this deep result, we instead include continuity in the definition.

In the following, we will for simplicity consider the case $n = 2$. The reader may easily formulate the analogous results for $n > 2$.

In the one-variable theory, very natural domains for investigating analytic functions are open discs:

$$D_r(a) = \{ z \in \mathbb{C} : |z - a| < r \}. \quad (2)$$

In two variables it can be natural to consider open balls:

$$B_r(a) = \{ z \in \mathbb{C}^2 : \|z - a\| < r \}, \quad (3)$$

where $a \in \mathbb{C}^2$, or simply

$$B_r = \{ z \in \mathbb{C}^2 : \|z\| < r \}. \quad (4)$$

But as it turns out, it can often be even more natural to consider so called polydiscs:

$$P(r, a) = D_{r_1}(a_1) \times D_{r_2}(a_2) = \{ z \in \mathbb{C}^2 : |z_1 - a_1| < r_1, |z_2 - a_2| < r_2 \}, \quad (5)$$

where $r = (r_1, r_2) \in \mathbb{R}^2$. Polydiscs can be thought of as sharing properties both with discs and squares. Just like other sets, polydiscs have a topological boundary $\partial P(r, a)$ (with real dimension equal to 3). But it is often more useful to consider the following subset of the boundary which is usually referred to as the distinguished boundary:

$$\partial_0 P(r, a) = \{ z \in \mathbb{C}^2 : |z_1 - a_1| = r_1, |z_2 - a_2| = r_2 \}, \quad (6)$$

with (real) dimension 2. If the polydisc is centered at the origin, we can use the notation

$$P(r_1, r_2) = \{ z \in \mathbb{C}^2 : |z_1| < r_1, |z_2| < r_2 \}, \quad (7)$$
2. The Cauchy Formula in Polydiscs

The following theorem is the easiest way to generalize the Cauchy Integral Formula to several variables:

**Theorem 1.** Suppose that \( f(z_1, z_2) \) is analytic in a neighbourhood of the closed polydisc \( \overline{P(r, a)} \). Then for all \( z \in P(r, a) \),

\[
f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{\partial_b P(r, a)} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2. \tag{8}
\]

The proof is simply a repeated application of the usual Cauchy Formula. From this formula, many of the standard properties of analytic functions follow immediately. For instance, by differentiating under the integral signs we see that analytic functions are infinitely differentiable and the derivatives are given by

\[
D^j D^k f(z_1, z_2) = \frac{j! k!}{(2\pi i)^2} \int_{\partial_b P(r, a)} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)^{j+1}(\zeta_2 - z_2)^{k+1}} d\zeta_1 d\zeta_2, \tag{9}
\]

where

\[
D^j D^k f(z_1, z_2) = \frac{\partial^{j+k} f}{\partial z_1^j \partial z_2^k}(z_1, z_2). \tag{10}
\]

We also get power series expansions very much like in the one-variable case. If we for simplicity restrict to the case of a polydisc centered at the origin, we get by geometric expansion

\[
f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{\partial_b P(r, 0)} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 = \tag{11}
\[
\frac{1}{(2\pi i)^2} \int_{\partial_b P(r, 0)} \frac{f(\zeta_1, \zeta_2)}{\zeta_1 \cdot \zeta_2 (1 - \frac{z_1}{\zeta_1})(1 - \frac{z_2}{\zeta_2})} d\zeta_1 d\zeta_2 = \tag{12}
\]

\[
\frac{1}{(2\pi i)^2} \int_{\partial_b P(r, 0)} \frac{f(\zeta_1, \zeta_2)}{\zeta_1 \cdot \zeta_2} \left( \sum_{j=0}^{\infty} \left( \frac{z_1}{\zeta_1} \right)^j \right) \left( \sum_{k=0}^{\infty} \left( \frac{z_2}{\zeta_2} \right)^k \right) d\zeta_1 d\zeta_2 = \tag{13}
\]

\[
= \sum_{j,k}^{\infty} a_{j,k} z_1^j z_2^k, \tag{14}
\]

where

\[
a_{j,k} = \frac{1}{(2\pi i)^2} \int_{\partial_b P(r, 0)} \frac{f(\zeta_1, \zeta_2)}{\zeta_1^{j+1} \zeta_2^{k+1}} d\zeta_1 d\zeta_2. \tag{15}
\]

By comparing with (9) above we also see that these coefficient \( a_{j,k} \) can in fact be expressed as derivatives of \( f \):

\[
a_{j,k} = \frac{1}{j! k!} \frac{\partial^{j+k} f}{\partial z_1^j \partial z_2^k}(0, 0). \tag{16}
\]

Thus the series expansion in (14) can indeed be viewed as a two-variable Taylor series.

3. Analytic Continuation

In the previous section we have seen that many properties of analytic functions of one complex variable carry over to several variables in a very natural way. This however, is not the full story; there are also completely new phenomena. The most important new aspect has to do with analytic continuation. Let us start by recalling the following theorem (which is so far exactly the same in one and several variables):
Theorem 2 (Principle of Analytic Continuation). Suppose that $\Omega_0 \subset \Omega$ are open (connected) domains in $\mathbb{C}$, $\mathbb{C}^2$ or $\mathbb{C}^n$, and let $f \in A(\Omega_0)$. If $F, G \in A(\Omega)$ both agree with $f$ in $\Omega_0$, i.e. $F(z) = f(z) = G(z)$ for all $z \in \Omega_0$, then $F(z) = G(z)$ for all $z \in \Omega$.

In other words, if an analytic function can be extended to a larger domain, then the extension is unique. We can therefore talk about the analytic continuation of $f$ to $\Omega$.

A standard proof in one variable is based on continuation along curves and power series expansion. And since the power series expansion in polydiscs of the previous section works equally well for this purpose, the proof translates word-by-word to the present case.

In one variable however, the possibility of an analytic continuation is something which essentially depends on the function itself: in any domain there are examples of functions which can and functions which cannot be extended to larger domains.

Not so in several variables. The following examples gives an example of a pair of domains $\Omega_0 \subset \Omega$ such that every function in $A(\Omega_0)$ extends to a function in $A(\Omega)$ (in a unique way).

Example 1 Let $\Omega_0 = \{z = (z_1, z_2) \in \mathbb{C}^2 : 1 < \|z\| < 3\} = B_3 \setminus B_1$. We will now prove that every function $f \in A(\Omega_0)$ extends to a function in the full ball $\Omega = B_3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : \|z\| < 3\}$. To this end, we first observe that we have the following inclusions:

$$B_1 \subset P(2,2) \subset B_3,$$

where $P(2,2)$ is the polydisc centered at the origin with radii $(2,2)$. In addition, the set $C = \{(z_1, z_2) : |z_1| < 2, |z_2| = 2\}$ is contained in $\Omega_0$, so the integral

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w| = 2} \frac{f(z_1, w)}{w - z_2} dw$$

is well-defined in $P(2,2)$, and furthermore we see that it is analytic there, essentially by differentiating under the integral sign. However, in the open subset $D = \{(z_1, z_2) : 1 < |z_1| < 2, |z_2| < 2\}$ (shaded in the figure), we can (since $f(z_1, z_2)$ for fixed $z_1, |z_1| > 1$ is analytic in the disc $|z_2| < 2$) apply the Cauchy Integral Formula to conclude that

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w| = 2} \frac{f(z_1, w)}{w - z_2} dw = f(z_1, z_2).$$

By the Principle of Analytic Continuation, $F$ and $f$ must agree on the connected set $\Omega_0 \cap P(2,2)$, and since $F$ is actually well-defined and analytic in a neighbourhood of $\overline{B_1}$, we have produced the required analytic continuation to all of $\Omega$. Note that since the polydisc is four-dimensional, it is not possible to draw it in a completely adequate way, hence in the figure it looks like a three-dimensional cylinder.

It is not difficult to extend the method of the example to conclude that any function, analytic in a region $\Omega = B_R \setminus B_r$ (where $0 < r < R$), automatically extends to a function
analytic in all of $B_R$. For more complicated domains, the method is unpractical and it is much more efficient to use methods from the theory of partial differential equations. However, such techniques go beyond the aim of these simple notes (see Hörmander [1]). It is in fact not so difficult to show that given any bounded domain $\Omega$ and any compact subset $K \subset \Omega$, one can extend an arbitrary analytic function in $\Omega \setminus K$ to all of $\Omega$. Thus, in a sense, “interior holes” in the domain of definition of the function can always be filled in, in a unique way.

But it can also be possible to fill in holes from the exterior side:

**Example 2** Let $\Omega_0 = \{ z = (z_1, z_2) \in \mathbb{C}^2 : \text{Re} \, z_1 < 0, \|z\| > 1 \} = H \setminus B_1$, where $H = \{ z : \text{Re} \, z_1 < 0 \}$. Then every function $f$ which is analytic in $\Omega$ extends to a function analytic in $\Omega = H$.

The idea is the same as in example 1. We let $C = \{ (z_1, z_2) : \text{Re} \, z_1 < 0, |z_1| < 2, |z_2| = 1 \}$ and observe that the function

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(z_1, w)}{w - z_2} \, dw$$

(20)

is well-defined and analytic in $D = \{ (z_1, z_2) : |z_1| < 2, |z_2| < 1 \}$ and furthermore coincides with $f$ whenever $|z_2| < 1$ and $\text{Re} \, z_1 < -1$, thus defines an analytic extension.

On the other hand, there are of course lots of domains from which it is not possible to extend all analytic functions, e.g. convex sets:

**Example 3** If $\Omega \subset \mathbb{C}^2$ is convex, then for every point $a \in \partial \Omega$, there is an analytic function which can not be extended to an analytic function in any domain containing $a$ as an interior point.

In fact, convexity implies that there is a hyperplane $V$ (real dimension 3) and normal vector $w$ such that $\Omega$ lies entirely on one side. The function

$$f(z) = \frac{1}{\langle z - a, w \rangle}$$

(21)

then gives an example of an analytic function in $\Omega$ which will tend to $\infty$ when $z \to a$ and hence can not be extended.

The class of convex domains is not the largest class of domains which do not allow analytic continuation independently of the function. The correct characterisation uses a slightly weaker concept called pseudo-convexity which in a sense is similar to ordinary convexity, but which also takes into account the complex structure. Again, we will not go into details but instead refer to Hörmander [1].

**References**

Problems

(1) Find the largest polydisc $P$ centered at the origin in which the function
\[ f(z, w) = \frac{z^2 w^3}{(1 - z)(1 - w)} \]
is analytic. Determine the power series expansion of $f(z, w)$ in $P$.

(2) Show, by means of an example, that in general there is no largest polydisc in which a function is analytic. (Hint: In what polydiscs is the following function analytic?)
\[ f(z, w) = \frac{1}{1 - zw} \]

(3) Prove that if $f(z)$ is analytic in the unit ball $U = \{ z \in \mathbb{C}^n : |z| < 1 \}$, then the power series expansion of $f(z)$ converges at every point in $U$.
More generally, prove that if $\Omega \subset \mathbb{C}^n$ is a union of (complex linear transformations of) polydiscs centered at the origin, and if $f(z)$ is analytic in $\Omega$, then the power series expansion of $f(z)$ at the origin converges at every point in $\Omega$.

(4) A function $u$ is said to be harmonic in $\Omega \subset \mathbb{C}^n$ if
\[ \sum_{k=1}^{n} \left( \frac{\partial^2 u}{\partial x_k^2} + \frac{\partial^2 u}{\partial y_k^2} \right) = 0, \]
where $z_k = x_k + iy_k$. $u$ is said to be pluriharmonic if
\[ \frac{\partial^2 u}{\partial x_k^2} + \frac{\partial^2 u}{\partial y_k^2} = 0, \quad \text{for all } k = 1, \ldots, n. \]
Give an example of a function $u$ in $\mathbb{C}^2$ which is harmonic but not pluriharmonic. (Compare with Hartog’s theorem which states that a function is analytic if and only if it is analytic in each variable separately.)

(5) Let $\partial_0 P = \{ (z, w) : |z| = |w| = 1 \}$ with the usual orientation. Compute the integral
\[ \iint_{\partial_0 P} \frac{dz \, dw}{3 - zw}. \]

(6) Let $\partial_0 P = \{ (z, w) : |z| = |w| = 1 \}$ with the usual orientation. Compute the integral
\[ \iint_{\partial_0 P} \frac{dz \, dw}{1 - 3zw}. \]