Combinatorics II (Online course, omtentamen)
August 20, 2014
9:00 - 14:00

SOLUTIONS

1. (3 points) Give the rook polynomial of a rectangular 6×13 'chessboard' where one of the corner tiles is removed (so the board has $6 \cdot 13 - 1$ many tiles). You must show your reasoning.

Solution:

Let C be the full 6×13 chessboard, C' the full 5×12 chessboard, and C'' the one we are looking for. Then we have $r(C, x) = xr(C', x) + r(C'', x)$. Since the rook polynomials of full chessboards are known (right?), solving for $r(C'', x)$ and simplifying yields

$$r(C'', x) = 1 + \sum_{k=1}^{6} \left(\binom{6}{k} \binom{13}{k} k! - \binom{5}{k-1} \binom{12}{k-1} (k-1)! \right) x^k.$$

2. (4 points) Find a closed formula for the sequence a_n that satisfies $a_0 = 0$, $a_1 = 67/5$, and for $n \geq 2$ the recursion relation

$$a_n - a_{n-1} - 6a_{n-2} = 30 \cdot n \cdot 3^{n-2}.$$

Clearly present every step of your computation.

Solution:

The characteristic roots are $3, -2$. The right side indicates to set up $a_n^{(p)} = n(An + B)3^n$ as a particular solution. This leads to $A = 1, B = 9/5$. Solving $a_n = c_1(-2)^n + c_2 3^n + n(n + 9/5)3^n$ for the initial conditions gives the solution

$$-(-2)^n + 3^n + n(n + 9/5)3^n.$$

3. (3 points) Let G be a complete 6-ary rooted tree with 131 leafs. How many vertices does G have?

You have to give a complete argument, don't just use some formula.

Solution:

Let s be the number of leafs. Since G is a tree we know

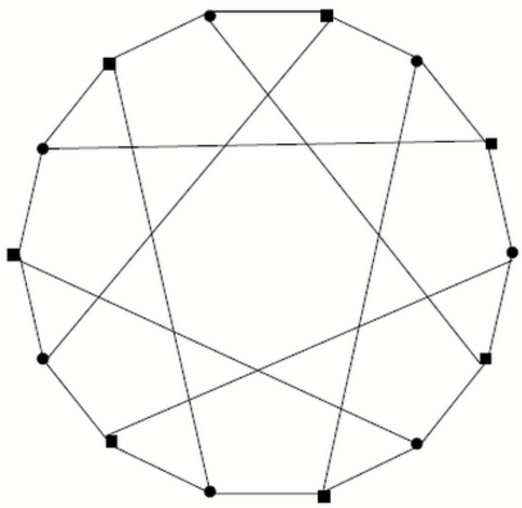
$$\sum_{v \in V} \deg(v) = 2|E| = 2(|V| - 1).$$

Since only a leaf has degree one, the root has degree 6, and every other vertex has degree $6 + 1 = 7$, the left side of the previous equation equals

$$s + 6 + 7(|V| - 1 - s).$$

Solving the equation with $s = 131$ yields $|V| = 157$. There are of course also other easier ways to solve this problem.

4. (3 points)
Prove that the following graph is not planar:



Hint: Kuratowski's theorem. Don't give up too early, this is something to play around with.

Solution:

To not spoil the fun, let's just say that one can find a subgraph (in many ways!) that is homemorphic to $K_{3,3}$.

Here's a different proof (that is slightly less rigorous). Assume that the graph is planar. Since the graph has 14 vertices and 21 edges, a planar drawing has 9 regions. One 'observes' that every cycle has at least 6 edges. Therefore, the sum over the degrees of the regions is at least 54. On the other hand, this number equals twice the number of edges, namely, 42, a contradiction.

5. (3 points) You probably heard about the following fact: every natural number n has a unique representation as a sum of nonnegative integers in base 3. This means that

$$n = c_0 + c_1 \cdot 3 + c_2 \cdot 3^2 + \cdots ,$$

where c_i is either 0, 1 or 2; and these numbers are unique. In this way, $100 = 1+0\cdot3+2\cdot9+1\cdot81$. We say that the decimal number 100 equals **1201** as a *ternary number*.

Here's your problem. How many ternary numbers of length 15 are there consisting of five 0's, five 1's, and five 2's as digits, if it is not allowed that the same digits all are next to each other? (So, not all five 0's are allowed to appear consecutively as digits, and the same holds for the five 1's, and for the five 2's). For instance, **222200010211110** is one of these numbers.

To make your life easier, the first digit is allowed to be 0, so **022220010211110** can be regarded as a ternary number of length 15.

Solution:

We use inclusion-exclusion, where c_i is the condition that all i's are next to each other (so appear as one block!). Therefore, we look for $S_0 - S_1 + S_2 - S_3$. Clearly, $S_0 = \frac{15!}{5!5!5!}$. If e.g.

0 appears as one block, then we have $\frac{(10+1)!}{1!5!5!}$ many such numbers, so we get $S_1 = 3\frac{11!}{5!5!}$. If e.g. 0 and 1 appear as one block, we have $\frac{(1+1+5)!}{1!1!5!}$ many such numbers, so $S_2 = \binom{3}{2}\frac{7!}{5!} = 3 \cdot 7 \cdot 6$. Finally, $S_3 = 3!$. The overall answer is 748560 (it wasn't necessary to evaluate it).

6. (1,5 points) Let a_r be the number of nonnegative integer solutions of the equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = r.$$

Write down *two* different interpretations of a_8 in terms of (unordered) partitions. Use precise language. No justifications necessary.

Solution:

a_8 is the number of unordered partitions of 8 into parts of size at most 5. By the Ferrers graph, it also equals the number of unordered partitions of 8 into at most 5 parts.

7. (3 points) 64 employees should be assigned to work on 18 projects. Each employee should work on exactly one project, and for every project at least 3 and at most 4 employees should be assigned. In how many ways can this be done?

As usual it is enough to write down an explicit formula, you don't have to evaluate it.

Solution:

Using exponential generating functions we see that the answer is the coefficient of $\frac{x^{64}}{64!}$ in

$$(\frac{x^3}{3!} + \frac{x^4}{4!})^{18}.$$

Hence, the answer is $\binom{18}{8}\frac{64!}{(3!)^8(4!)^{10}}$.

8. (3 points) In a candy factory, surprise boxes containing five types of godis are manufactured. Here the following rules must be followed:

- the number of cola bottles in a box must be divisible by 4,
- the number of slickepinnar in a box must be even,
- the square root of the number of vingummi must be an even integer,
- there cannot be more than three praliner in a box,
- there cannot be more than one chocoladkaka in a box,
- the total number of godis is between 17 and 25.

As you see it is possible that a box might have only one type of godis. Now, during one production cycle every possible kind of box gets produced one after another without ever repeating an assortment. If each assorting of a box takes 5 seconds and you would like to have a box containing a vingummi, what's the longest time you would possibly have to wait?

Solution:

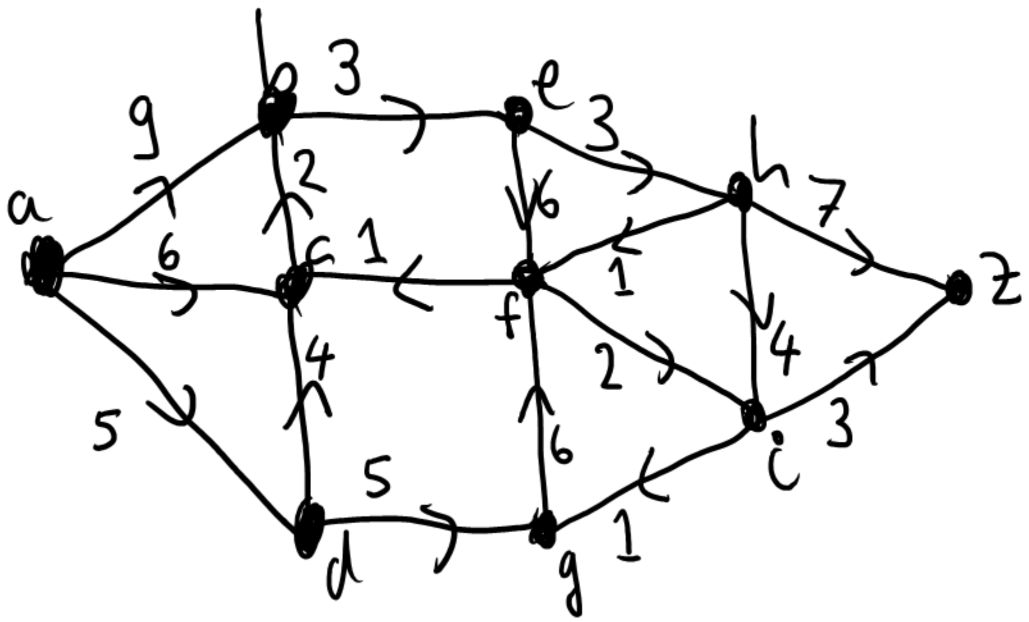
We have to find out the number of possible boxes containing no vingummi. Using generating functions, we see that this number equals the sum of the coefficients of $x^{17}, \dots, x^{25}$ in

$$(1 + x^4 + x^8 + \cdots)(1 + x^2 + x^4 + x^6 \cdots)(1 + x + x^2 + x^3)(1 + x).$$

Simplifying this generating function yields $1/(1-x)^2 = \sum_{r=0}^{\infty}(r+1)x^r$. Therefore, there are $18 + \cdots + 26 = 198$ many such boxes without vingummi. Therefore, we have to wait at most

$5 \cdot 198 = 990$ many seconds for all these boxes to have been produced, and then 5 seconds more for the production of the first with a vingummi (thanks to one student for pointing this out in her/his solution!). So, the answer is 995 seconds, a little more than a quarter of an hour.

9. (6,5 points) Consider the following network:



(a) (2,5 points) Use Dijkstra's algorithm to find for any vertex $v = a, b, c, d, e, f, g, h, i, z$ the distance $d(a, v)$ from a to v.

You MUST USE Dijkstra's algorithm in order to get any credits. Clearly show your table and in which order you choose the vertices (use an extra page, since you will need some space).

Solution:

Here is the table:

b	9	9	<u>8</u>						
c	6	<u>6</u>							
d	<u>5</u>								
e	∞	∞	∞	11	<u>11</u>				
f	∞	∞	∞	∞	16	16	<u>15</u>		
g	∞	10	10	<u>10</u>					
h	∞	∞	∞	∞	∞	<u>14</u>			
i	∞	∞	∞	∞	∞	∞	18	<u>17</u>	
z	∞	∞	∞	∞	∞	∞	21	21	<u>20</u>

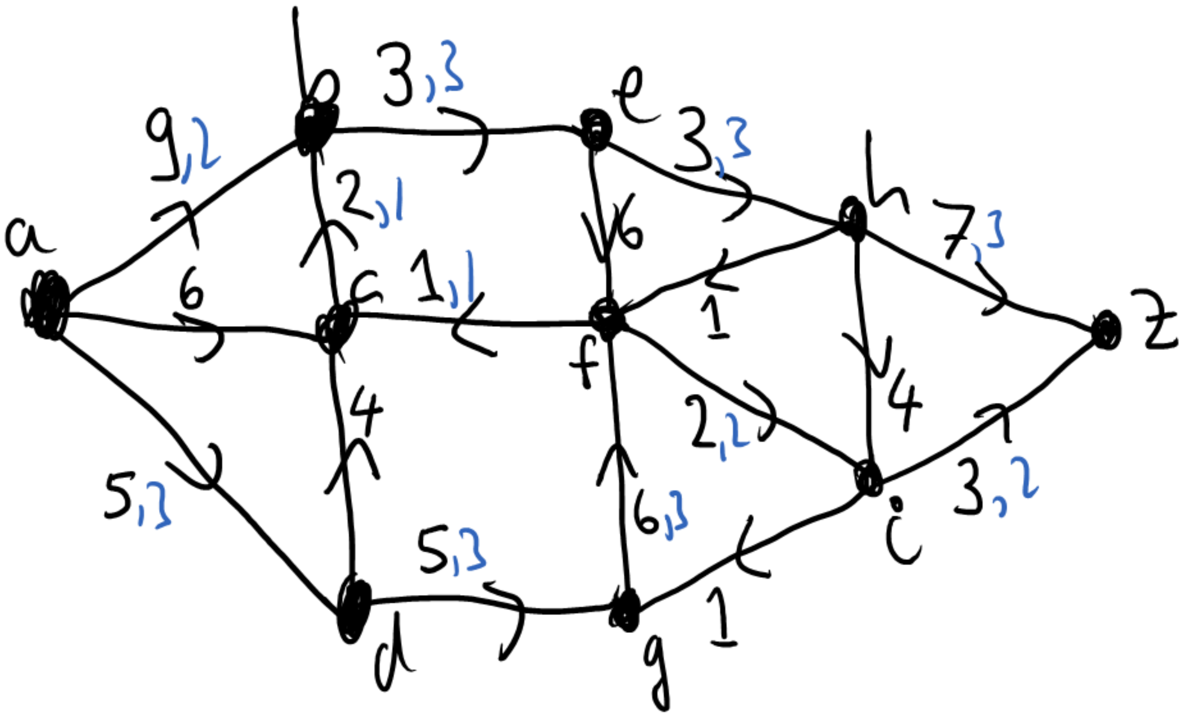
(b) (1 points) Give a shortest directed path from a to z.

Solution:

a, c, b, e, h, f, i, z.

(c) (2 points) Give a flow with maximal flow value where (to make things more interesting) *the arrow from f to c has flow value* 1.

Enter the flow you found RIGHT HERE into the network (next to the capacities):



Write here the flow value of your flow:

Solution:

5.

(d) (1 points) Give a cut with the minimal cut capacity RIGHT HERE:
Solution:

$P = \{a, b, c, d, e, f, g, h\},\ \bar{P} = \{h, i, z\}$.

Write here the cut capacity of your cut:

Solution:

5.