MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II (Online course; omtentamen)
August 26, 2015
9:00 - 14:00

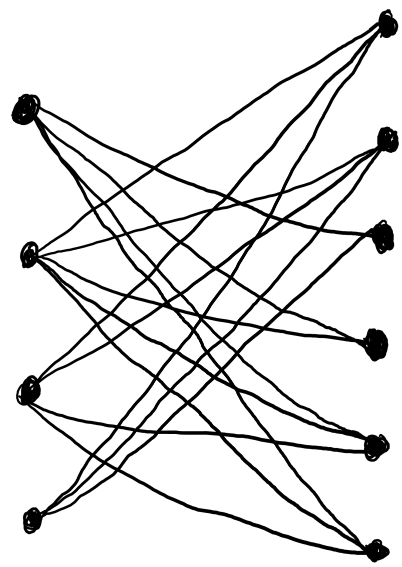
The exams can be picked up from Katarina Ringels (Hus 6, Rum 204).

Complete and clear solutions must be given
except where explicitly stated otherwise.

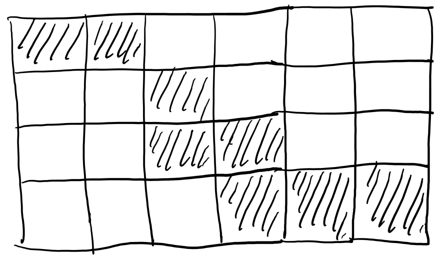
No calculators allowed.
If you cannot simplify an expression any further, just leave it.

1. (3 points) Prove that, if A is any set of eight numbers selected from the set $\{1, \dots, 20\}$, then there are two different 4-subsets of A that have the same sum.
Remember that a 4-subset has four elements. For instance, for $A = \{1, 2, 3, 5, 7, 10, 11, 20\}$, we can choose $\{1, 2, 11, 20\}$ and $\{1, 3, 10, 20\}$.

2. (4 points) Consider the following bipartite graph.



We are interested in finding the number of complete matchings. I claim you can do this by using that $1 + 8x + 21x^2 + 20x^3 + 4x^4$ is the rook polynomial of the following **shaded** board (you don't have to prove that the rook polynomial is correct):



It is enough to give an explicit expression. You won't get points by simply counting all of the complete matchings. You **must** explain how to do it by using the given rook polynomial.

3. (3 points)
Find a closed formula for the sequence a_n that satisfies $a_0 = -3$ and for $n \geq 1$ the recursion relation

$$a_n = a_{n-1} + 2n - 2.$$

Clearly present every step of your computation.

4. (3 points) Little Gustav likes to play with a toy cone on which he stacks rings. How many possibilities for putting 10 rings on the cone are there, if

- the rings have 8 different colors,
- Gustav has only 3 of the yellow ones,
- but he has 12 rings of each of the other seven colors,
- and he always uses an even number of rings for each color.

It is enough to give an explicit expression.

5. (3,5 points) Let a_n be the number of possibilities to put at most n identical balls into 5 identical urns. Write the generating function of a_n as a quotient of two polynomials.

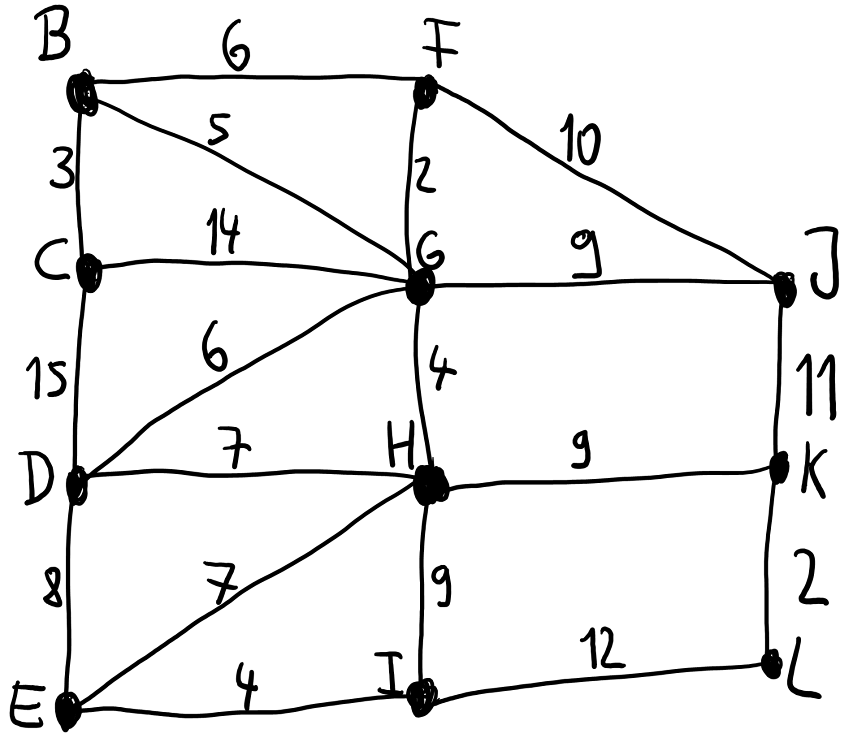
6. (3 points) Let G be a connected, loop-free, planar graph with 15 edges. Show that if G has no cycles of length strictly less than 5, then G has at least 11 vertices.

7. (2,5 points) A salesman visits by car all ten cities of a little state (including his own town) and wants to return home without visiting a town twice. He knows that the capital is the only town that has a direct road connection to any other town. If he knows that overall there are 39 direct road connections between the cities, why can he be sure that there is such a tour?

Hints: If you don't know how to do it, explain your ideas. How can you translate it into a graph problem? What is the name of such a tour? How many direct road connections does a city at least have?

GO TO THE NEXT PAGE.

8. (3 points)
It is planned to connect 11 small towns by gas pipelines, where the gas is produced in F. The pipeline between F and G had already been built. In following diagram you see how expensive it would be to build the pipelines. Here, e.g. 2 stands for 2,000,000 SEK.



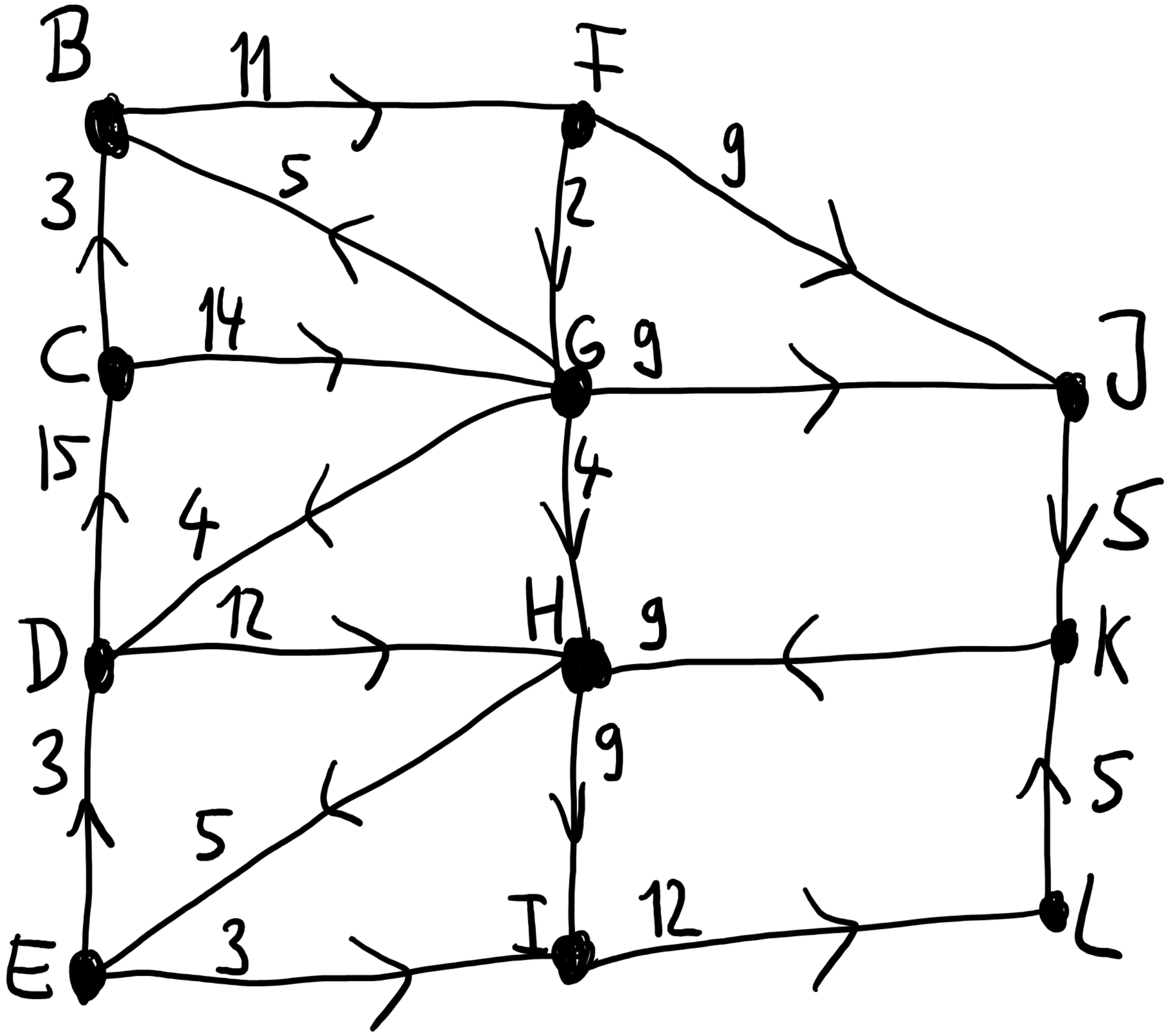
(a) (2 points) Use **Prim's algorithm** to determine which additional pipelines would have to be built in order to connect by them all 11 cities while keeping the construction costs as low as possible.
(b) (1 points) What are the minimal construction costs?

GO TO THE NEXT PAGE.

9. (5 points)

(a) (4 points) At factories B,C,D,E computer chips are manufactured and are then transported to computer manufacturers J,K,L. The capacities of the arrows in the following graph denote how many units can be transported that way. Now, Factory B produces at most 3 units, factory C at most 13 units, D at most 3 units, and factory E produces at most 13 units. The manufacturer J can only receive at most 13 units, K at most 15 units, and manufacturer L can only take at most 14 units. Find the maximal possible number of chips that can be produced and transported.

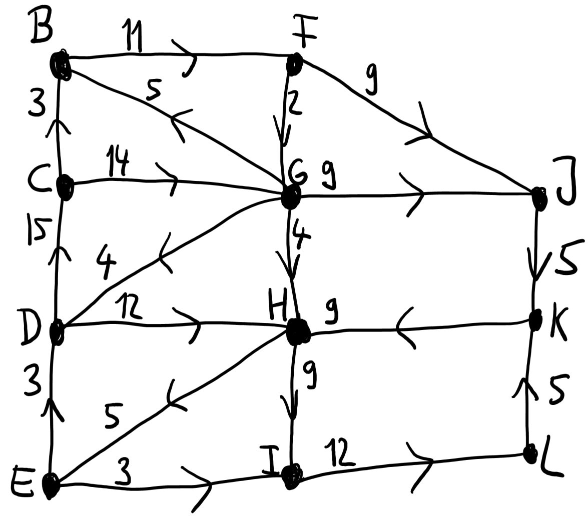
Change the graph first to get a network. You must use the Ford-Fulkerson and Edmonds-Karp algorithms.

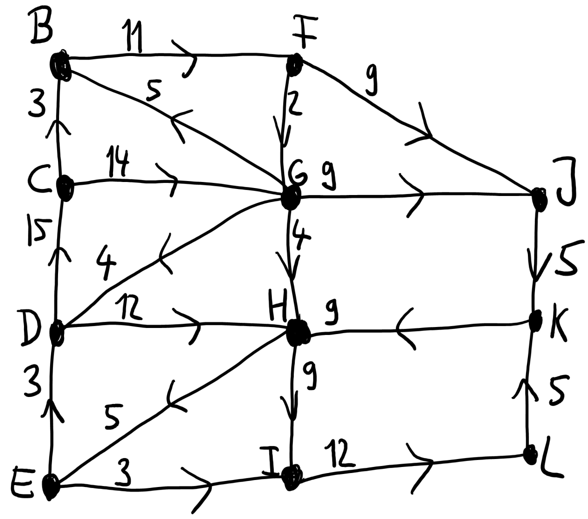


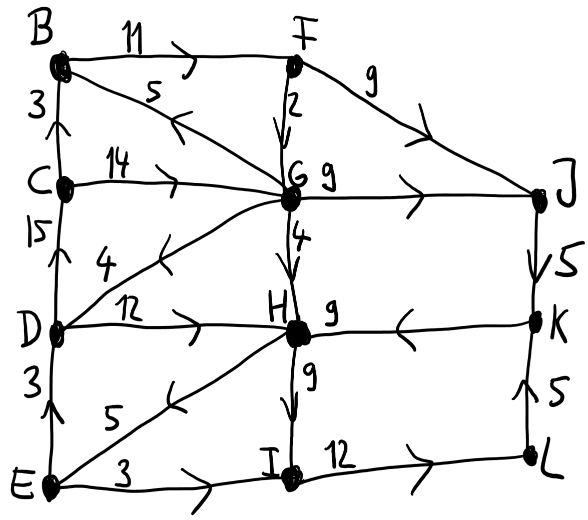
*****DO THIS ON THE NEXT FEW PAGES*****

HAND IN THIS PAGE

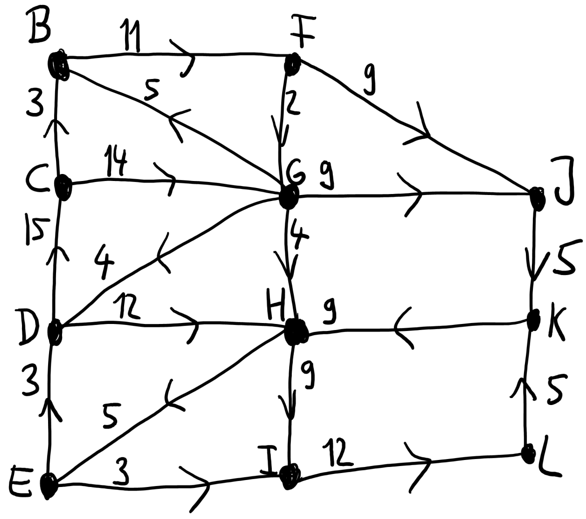
Draw after each iteration in the algorithm the improved flow RIGHT HERE into the network (next to the capacities). Indicate when you have found a maximal flow. (You may not need all diagrams.)

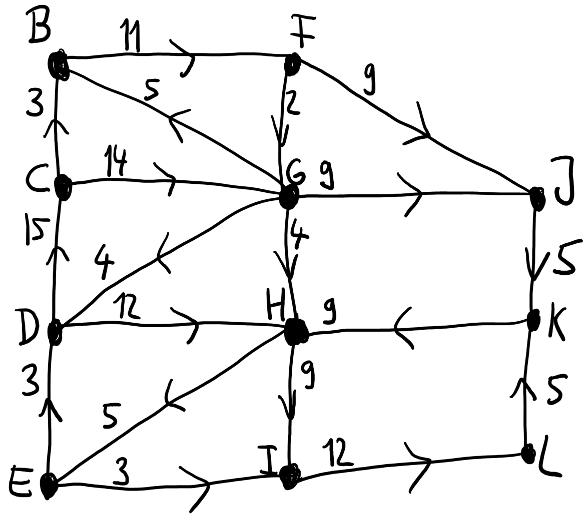


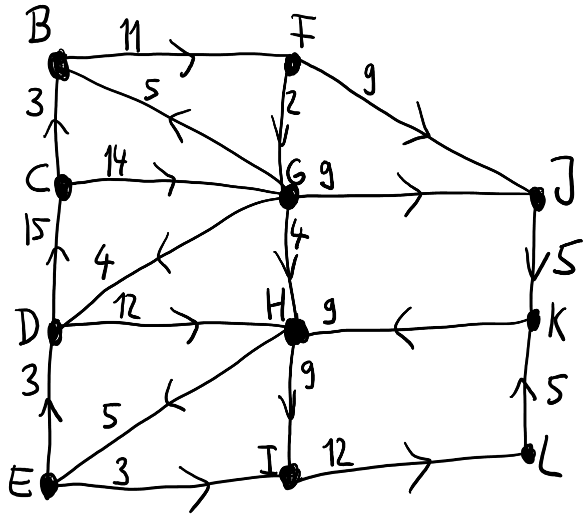




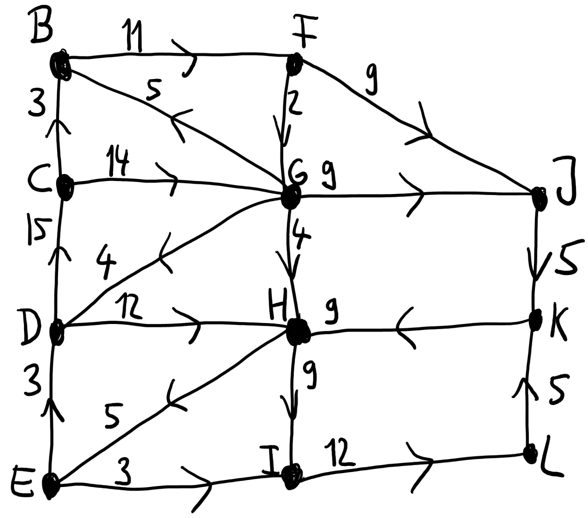
*****HAND IN THIS PAGE*****

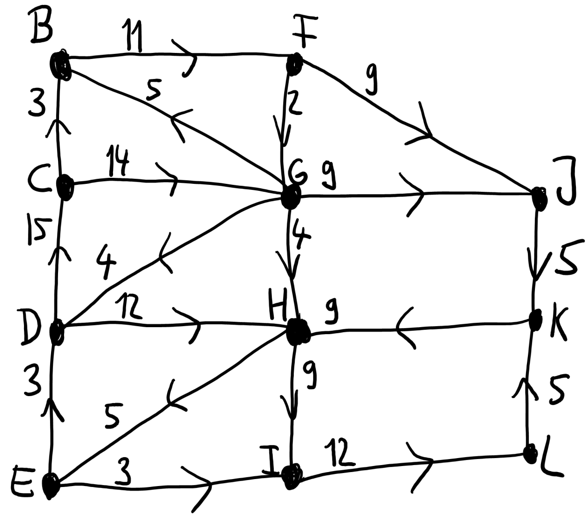


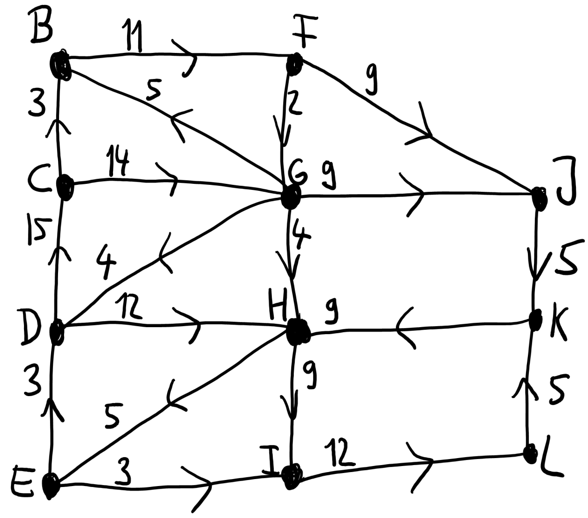




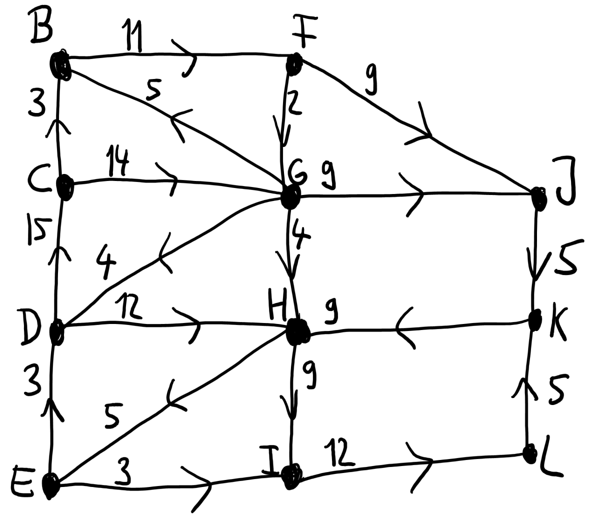
HAND IN THIS PAGE

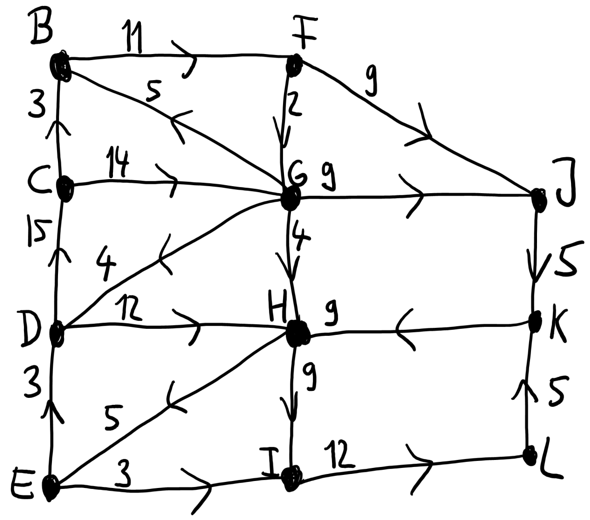






HAND IN THIS PAGE





Write here the number of chips that are transported according to your solution:

(b) (1 points) Prove your result is true by giving a cut with the minimal cut capacity RIGHT HERE that corresponds to the output of the algorithm above:

Write here the cut capacity of your cut: