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Exam / Tentamen
MM7022 Logik II, 7,5 hp
HT 2021
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This exam consists of 6 questions, worth a total of 40 points. Not all questions are equally difficult. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

Good luck! — Lycka till!

1 Calculate the cardinalities of the following sets, and order them according to cardinality (some may be equal):

- $S_1 \coloneqq \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ increasing}\}$
- $S_2 \coloneqq \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ decreasing}\}$
- $S_3 \coloneqq \mathbb{N}^{\mathbb{R}}$
- $S_4 \coloneqq \{f : \mathbb{R} \to \mathbb{N} \mid f \text{ increasing}\}$

(Here "increasing" means "non-strictly increasing", i.e. $f(x) \leq f(y)$ *whenever* $x \leq y$*, and similarly "decreasing" means "non-strictly decreasing".)*

2 Work in ZF. The *order extension principle* is the statement "for every partially ordered set $(X, \leq)$, there is some total order $\leq'$ on X with $(\leq) \subseteq (\leq')$," or more concisely "every partial order can be extended to a total order".

(a) Show that AC implies the order extension principle. *(You may use AC either directly, or via any of its consequences considered in the course.)*

(b) Show that the order extension principle implies the restricted form of AC, "for every family of non-empty *finite* sets, there exists some choice function".

3 Cardinalities of models.

(a) Show that any consistent theory T in a language L has some model of size $\leq \max(\|T\|, \|L\|)$.

(b) Give an example of a consistent theory T in a language L with no model of size $< \max(\|T\|, \|L\|)$.

(c) Must a consistent theory T in a language L have some model of size exactly $\max(\|T\|, \|L\|)$? Prove or give a counterexample.

(d) Must a consistent theory T in a language L have some model of size $\leq \|T\|$, assuming $\|T\|$ is infinite? Prove or give a counterexample.

4 Show that the following functions are recursive:

(a) "truncated predecessor" $p : \mathbb{N} \to \mathbb{N}$, given by $p(0) := 0$, $p(n) := n - 1$ for $n > 0$.

(b) "truncated subtraction", aka "monus" $\dot{-} : \mathbb{N}^2 \to \mathbb{N}$, $m \dot{-} n := \max(m - n, 0)$.

(Recall that the partial recursive functions are generated by projections, the constant 0, successor, composition, primitive recursive definitions, and the (unbounded) minimisation operator.)

5 ZFC is a theory in the countable language $\langle \in \rangle$. So by the Löwenheim–Skolem theorems, if ZFC is consistent, it has some countable model $\mathcal{M}$. But ZFC proves "there exists an uncountable set", so this statement holds in $\mathcal{M}$. Why doesn't this give a contradiction?

6 The goal of this problem is to show there are uncountably many countable non-standard models of PA, up to isomorphism. Work over the language of arithmetic $L_A := \langle 0, 1, S, +, \times \rangle$. For $n \in \mathbb{N}$, write $\bar{n}$ for the term $S^n(0)$ in L_A. In L_A, define "$x \geq y$" as "$\exists z\, x = y + z$" and "$x \mid y$" as "$\exists z\, y = z \times x$" A model of PA is *standard* if every element is the interpretation of some $\bar{n}$; equivalently, if it is isomorphic to the model $\langle \mathbb{N} ; 0, 1, S, +, \times \rangle$.

(a) Consider L_A plus an extra constant symbol c; let T be the theory consisting of PA, together with the axiom "$c \geq \bar{n}$" for each $n \in \mathbb{N}$. Show that T is consistent; deduce that there exists some non-standard model of PA.

(b) Write $\mathbb{P} \subseteq \mathbb{N}$ for the set of prime numbers. Show that for every set of primes $X \subseteq \mathbb{P}$, there is some countable model $\mathcal{M}$ of PA with an element $a \in \mathcal{M}$ whose standard prime divisors are precisely X; that is, such that for each $p \in \mathbb{P}$, $\mathcal{M} \models \bar{p} \mid a$ if and only if $p \in X$.

(c) Deduce that there are uncountably many isomorphism classes of countable models of PA. *(Hint: note that for each such model, only countably many sets of primes can appear as sets of standard prime divisors of some element.)*

——— End of exam — Slut på provet ———