MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sara Maad Sasane

Tentamensskrivning i
Linjär analys
7,5 hp
2010-01-11

No calculator is allowed. The problems are not arranged in order of difficulty. Result from a subproblem may be used in subsequent subproblem even if the earlier one is not attempted. Justify your answers. Grade criteria A: 90–100 points, B: 75–89 points, C: 65–74 points, D: 55–64 points, E: 50–54 points, Fx: 40–49 points, F: 0–39 points. For passing a minimum grade of E is necessary in the written exam and moreover a pass in the compulsory assessed coursework.
Good luck!

1. Let $f : [0, \pi] \to \mathbb{R}$ be the function defined by $f(x) = 1 - x/\pi$.

   a) Determine the cosine series of the function f on the interval $(0, \pi)$. 10 p

   b) Show that the determined series in a) converges pointwise to f on $[0, \pi]$. Does it also converge uniformly on this interval? Justify your answer. 6 p

   c) The series from part a) converges on the whole of $\mathbb{R}$ to a function F, where $F(x) = f(x)$ when $x \in [0, \pi]$. Draw the graph of F on the interval $[-5\pi, 5\pi]$. 4 p

   d) Determine

   $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

   3 p

2. Consider the function f which is defined on the interval $[-\pi, \pi]$ and is given by

   $$f(x) = e^x + e^{-x}, \qquad x \in (-\pi, \pi).$$

   The function has the Fourier series

   $$f(x) \equiv \frac{e^{\pi} - e^{-\pi}}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\cos(nx)\right) \qquad -\pi < x < \pi.$$

   (You don't have to prove this.)

   a) Use a theorem about differentiation of Fourier series to determine the Fourier series of f' on the interval $(-\pi, \pi)$. Justify why the theorem is applicable. 10 p

   b) Draw the graphs of f and f' on the interval $(-\pi, \pi)$. 5 p

   c) Note that $f''(x) = f(x)$ on $(-\pi, \pi)$ but that the series that is obtained by termwise differentiation of the Fourier series of f' is not the same as the Fourier series of f. Explain why. 5 p

3. Let f be defined by

   $$f(x) = \begin{cases} x & \text{for } x \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

VAR GOD VÄND!

a) Express f as a Fourier sine integral, i.e. find $B(\alpha)$ such that

$$f(x) = \int_0^{\infty} B(\alpha)\sin(\alpha x)\,d\alpha$$

if $x \neq \pm 1$. 10 p

b) What is the value of the integral in part a) when $x = \pm 1$? 5 p

4. It can be shown that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(nx)$$

converges for every $x \in \mathbb{R}$. (You don't have to show this.) Show that this series is not the Fourier series of any piecewise continuous function f. *Hint: Use Bessel's inequality.* 12 p

5. Consider the heat equation $u_t = k u_{xx}$ for $x \in (0, \pi)$ and $t \geq 0$, with the boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0, \qquad t \geq 0,$$

and the initial condition

$$u(x,0) = \sin^4(x), \qquad x \in (0,\pi).$$

a) Solve the given boundary value problem. *Hint: You may use the trigonometric formula*

$$\sin^4(x) = \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

without proof. 10 p

b) What is $\lim_{t\to\infty} u(x,t)$? Interpret this result when u denotes the temperature in an infinite slab $0 < x < \pi$, $-\infty < y < \infty$ with insulated faces. 10 p

6. Let $u(\rho, \phi)$ denote the equilibrium temperature in a thin disk described by $\rho \leq 1$ in polar coordinates. The boundary of the disk $\rho = 1$ is kept at constant temperature $f(\phi)$. The function u is bounded in the disk and satisfies the Laplace equation which in polar coordinates is given by

$$\rho^2 u_{\rho\rho}(\rho,\phi) + \rho u_\rho(\rho,\phi) + u_{\phi\phi}(\rho,\phi) = 0, \qquad 0 < \rho < 1,\ -\pi < \phi < \pi,$$

where $u(1,\phi) = f(\phi)$ and $u(\rho,-\pi) = u(\rho,\pi)$, $u_\phi(\rho,-\pi) = u_\phi(\rho,\pi)$.

Use separation of variables to solve this boundary value problem. *Hint: The substitution $\rho = e^s$ can be used for solving one of the equations obtained after separating variables.* 15 p

The marked exams can be collected from Reine Elfsö in room 208, house 6.