Solutions to the examination in Fourier analysis, October 22, 2008

1. a) Find the Fourier series for

$$f(x) = \mid \sin\ x \mid, \ \mid x \mid < \pi.$$

Since f is even we need to consider the cosine series $f(x) = \frac{a_0}{2} + \sum a_n \cos nx$, where

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} \mid \sin x \mid \cos nx dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx = \frac{1}{\pi}\int_0^{\pi} (\sin(n{+}1)x{+}\sin(1{-}n)x)dx =$$

$$= \frac{1}{\pi}\left\{\frac{-\cos(n+1)x}{n+1} \mid_0^{\pi} - \frac{\cos(1-n)x}{1-n} \mid_0^{\pi}\right\} = \begin{cases} 0, \text{ if } n \text{ is odd} \\ \frac{4}{\pi(1-4k^2)}, \text{ if } n \text{ is even} = 2k. \end{cases}$$

Answer: $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos 2kx}{1-4k^2}$.

1b) Calculate the sum: $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$. If $x = 0$ then one gets $0 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(1-4n^2)}$ or, equivalently, $\sum_1^{\infty} \frac{1}{(4n^2-1)} = \frac{1}{2}$.

1c) Calculate the sum: $1 + \sum_{n=1}^{\infty} \frac{4}{(4n^2-1)^2}$. By Parceval's identity one has:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \quad \text{or}$$

$$\frac{1}{2}(\frac{4}{\pi})^2 + (\frac{4}{\pi})^2 \sum_1^{\infty} \frac{1}{(4n^2-1)^2} = \frac{4}{\pi^2}(2 + \sum_1^{\infty} \frac{1}{(4n^2-1)^2}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x dx;$$

$$2 + \sum_1^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi}{4}\int_{-\pi}^{\pi} \sin^2 x dx = \frac{\pi}{4}\int_0^{\pi} (1 - \cos 2x)dx = \frac{\pi^2}{4}.$$

Answer: $1 + \sum_1^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2}{4} - 1$.

2. a) Solve the boundary value problem:

$$\begin{cases} u_t = a^2 u_{xx} - \beta u \\ u_x(0,t) = u_x(l,t) = 0, \ \ u(x,0) = \phi(x). \end{cases}$$

Hint. $u = y(t)U(x,t)$.

One can separate the variables direct or use the substitution $u = y(t)U(x,t)$. I the latter case one gets $y'U + yU_t = a^2 yU_{xx} - \beta yU$ plus the boundary conditions. We then get two problems:

$$\begin{cases} U_t = a^2 U_{xx} \\ U_X(0,t) = U_x(l,t) = 0 \qquad \text{and } y' = -\beta y. \\ U(x,0) = \frac{\phi(x)}{y(0)}. \end{cases}$$

One can choose $y(t) = e^{-\beta t}$ and obtain $y(0) = 1$. Applying the usual technique we get

$$U(x,t) = \frac{a_0}{2} + \sum_1^{\infty} a_n e^{-(\frac{\pi a n}{l})^2 t} \cos\frac{\pi n x}{l},$$

where $a_n = \frac{2}{l}\int_0^l \phi(x) \cos\frac{\pi n x}{l} dx$.

Answer: $u(x,t) = e^{-\beta t}(\frac{a_0}{2} + \sum_1^{\infty} a_n e^{-(\frac{\pi a n}{l})^2 t} \cos\frac{\pi n x}{l})$.

b) Find the solution for $\phi(x) = \sin\ \frac{\pi x}{l}$.

If $\phi(x) = \sin \frac{\pi x}{l}$ then

$$a_k = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi k x}{l}dx = \frac{1}{l}\{-\cos\frac{\pi(k+1)x}{l}\frac{l}{\pi(k+1)}\mid_0^l - \cos\frac{\pi(1-k)x}{l}\frac{l}{\pi(1-k)}\mid_0^l\} =$$

$$= \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{4}{\pi(1-4n^2)} & \text{if } k = 2n. \end{cases}$$

3. Solve

$$\begin{cases} u_{tt} = u_{xx} \\ u(-1,t) = u(1,t) = u_t(x,0) = 0. \end{cases}$$

$$\text{where } u(x,0) = \begin{cases} 1 - 2 \mid x \mid, & \text{om } \mid x \mid \le \frac{1}{2} \\ 0, & \text{om } \frac{1}{2} \le \mid x \mid \le 1. \end{cases}$$

a) by using separation of variables; b) using d'Alembert's formula.

3 a). Using separation of variables one gets

$$\begin{cases} X'' + \lambda X = 0 \\ X(-1) = X(1) = 0 \end{cases}$$

and

$$\begin{cases} T'' + \lambda X = 0 \\ T'(0) = 0 \end{cases}.$$

The first Sturm-Liouville problem has two sequences of eigenvalues and eigenfunctions. Namely, substituting $X = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$ we get the system

$$\begin{cases} a\cos\sqrt{\lambda} + b\sin\sqrt{\lambda} = 0 \\ a\cos\sqrt{\lambda} - b\sin\sqrt{\lambda} = 0 \end{cases}$$

which has two types of non-trivial solutions

$$i)\cos\sqrt{\lambda} = 0; b = 0 \quad \text{and} \quad ii)\sin\sqrt{\lambda} = 0; a = 0.$$

Therefore we get i) $\alpha_n = \sqrt{\lambda_n} = \frac{(2n-1)\pi}{2}$; $X_n = \frac{\cos(2n-1)\pi x}{2}$ and ii) $\tilde{\alpha}_m = \sqrt{\widetilde{\lambda_m}} = \pi m$; $\widetilde{X}_m = \sin\pi m$.

Corresponding solutions to the second problem are $T_n = \frac{\cos(2n-1)\pi t}{2}$ and $\tilde{T}_m = \cos\pi m t$. Using the superposition principle we get

$$u(x,t) = \sum_{m=1}^{\infty} a_m \sin\pi m x\cos\pi m t + \sum_{n=0}^{\infty} b_n\cos\frac{(2n-1)\pi x}{2}\cos\frac{(2n-1)\pi t}{2}.$$

Since $u(x,0)$ is even then all a_m vanish and we get $\sum b_n \cos\frac{(2n-1)\pi x}{2} = u(x,0)$. Therefore

$b_n = \int_{-1}^{1}(1-2\mid x\mid)\cos\frac{(2n-1)\pi x}{2}dx = 2\int_0^{1/2}(1-2x)\cos\frac{(2n-1)\pi x}{2}dx = \frac{16(1-\cos(2n-1)\frac{\pi}{4})}{\pi^2(2n-1)^2}$.

3b). Using d'Alembert's formula we get $u(x,t) = \frac{1}{2}(F(x+t) + F(x-t))$ where $F(z)$ is the odd periodic extension of $u(x,0)$, see picture below:

4. a) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem:

$$\begin{cases} xX'' + \frac{1}{2}X' + \lambda X = 0 \text{ on } \quad [0,1] \\ X(0) = X'(1) = 0. \end{cases}$$

Hint. $x = t^2$.

One has to express the derivatives w.r.t x in terms of the derivatives w.r.t. t:

$$\frac{dX}{dx} = \frac{dX}{dt}\frac{dt}{dx} = \frac{dX}{dt}\frac{1}{(2\sqrt{x})} = \frac{1}{2t}\frac{dX}{dt}.$$

$$\frac{d^2X}{dx^2} = \frac{d}{dx}\left(\frac{dX}{dx}\right) = \frac{d}{dt}\left(\frac{dX}{dt}\frac{1}{2t}\right)\frac{dt}{dx} = \frac{d^2X}{dt^2}\left(\frac{1}{2\sqrt{(x)}}\right)^2 + \frac{dX}{dt}\frac{d}{dt}\left(\frac{1}{2t}\right)\frac{dt}{dx} =$$

$$= \frac{1}{4x}\frac{d^2X}{dt^2} - \frac{1}{4x^{3/2}}\frac{dX}{dt}.$$

Thus the above differential equation has the following form w.r.t the variable t :

$$x\left(\frac{d^2X}{dt^2}\frac{1}{4x} - \frac{dX}{dt}\frac{1}{4x^{3/2}}\right) + \frac{1}{4\sqrt{x}}\frac{dX}{dt} + \lambda X = 0$$

with the boundary conditions $X(0) = X'(1) = 0$. In other words we get the boundary value problem:

$$\begin{cases} \frac{d^2X}{dt^2} + 4\lambda X = 0 \\ X(0) = X'(1) = 0 \end{cases}.$$

The general solution of the differential equation is $X(t) = a\sin 2\sqrt{\lambda}t + b\cos 2\sqrt{\lambda}t$. Since $X(0) = X'(1) = 0$ one gets $a = 0$ and $\cos 2\sqrt{\lambda} = 0$ implying $2\sqrt{\lambda_n} = \frac{\pi}{2} + \pi n$ or $\sqrt{\lambda_n} = \frac{\pi+2\pi n}{4}$.

Answer. The eigenvalues are: $\frac{\pi}{2} + \pi n$, and the eigenfunctions are: $X_n = \sin\frac{(2n+1)\pi\sqrt{x}}{2}$.

b) Formulate the orthogonality property for the eigenfunctions and check it directly.

Rewriting the differential equation in the form

$$(\sqrt{x}X')' + \frac{\lambda}{\sqrt{x}}X = 0$$

we get that the weight in the orthogonality condition equals $\frac{1}{\sqrt{x}}$.

Check:

$$\int_0^1 \frac{1}{\sqrt{x}}\sin\frac{(2k+1)\pi\sqrt{x}}{2}\sin\frac{(2m+1)\pi\sqrt{x}}{2}dx = \int_0^1 \sin\frac{(2k+1)\pi t}{2}\sin\frac{(2m+1)\pi t}{2}dt = 0.$$

5. Express the function $f(x) = e^{-a|x|}\cos bx, \ a > 0$ as the Fourier integral.

The function is even and therefore one should use the cosine integral

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x dx, \text{ where } A(\alpha) = \frac{2}{\pi}\int_0^\infty f(x)\cos\alpha x dx.$$

Further

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty e^{-ax}\cos bx\cos\alpha x dx = \frac{1}{\pi}\int_0^\infty e^{-ax}[\cos(b-\alpha)x + \cos(b+\alpha)x]dx =$$

$$= \frac{1}{\pi}\left(\frac{-a\cos(b-\alpha)x + (b-\alpha)\sin(b-\alpha)x}{a^2+(b-\alpha)^2}e^{-ax} + \frac{-a\cos(b+\alpha)x + (b+\alpha)\sin(b+\alpha)x}{a^2+(b+\alpha)^2}e^{-ax}\right)\Big|_0^\infty$$

$$= \frac{1}{\pi}\left(\frac{a}{a^2+(b-\alpha)^2} + \frac{a}{a^2+(b+\alpha)^2}\right) = \frac{a}{\pi}\left(\frac{1}{a^2+(b-\alpha)^2} + \frac{1}{a^2+(b+\alpha)^2}\right).$$

6. a) Show that if f is continuos and 2π-periodic, and with continuous first derivative then its Fourier series converges absolutely.

Proof. Since f has a continuous derivative then we get

$$f' = \sum_{n=1}^{\infty}(nb_n \cos(nx) - na_n \sin(nx).$$

Parceval's identity implies that

$$\sum_{n=1}[(nb_n)^2 + (na_n)^2] = \frac{1}{\pi}\int_{-\pi}^{\pi}(f'(x))^2 dx < \infty \Rightarrow \text{ by Cauchy's inequality}$$

$$\sum_{n=1}^{\infty} \mid a_n \mid + \mid b_n \mid \leq (\sum_{n=1}^{\infty} 1/n^2)^{1/2} (\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2))^{1/2} < \infty,$$

i.e. the Fourier series is absolutely convergent.

b) Formulate the uniqueness theorem for solutions of the wave equation.

See Churchill-Brown Ch. 11, §98.