Solutions to Fourier analys, January 12, 2008

1. a) Find the sine series for

$$f(x) = \begin{cases} \frac{cx}{\alpha}, & \text{where } 0 < x < \alpha \\ c, & \text{where } \alpha < x < \pi - \alpha \\ \frac{c(\pi - x)}{\alpha}, & \text{where } \pi - \alpha < x < \pi. \end{cases}$$

One has $f(x) = \sum_1^\infty b_n \sin nx$ where

$$b_n = \frac{2}{\pi}\Big\{\int_0^\alpha \frac{cx\sin nx}{\alpha}dx + \int_\alpha^{\pi-\alpha} c\sin nx dx + \int_{\pi-\alpha}^{\pi} \frac{c(\pi - x)\sin nx}{\alpha}dx\Big\}.$$

It is easy to calculate $\int x\sin n(x-\gamma)dx = -\frac{x\cos n(x-\gamma)}{n} + \frac{\sin n(x-\gamma)}{n^2}$.
Therefore,

$$b_n = \frac{2c}{\pi}\Big\{\frac{1}{\alpha}[-\alpha\frac{\cos n\alpha}{n} + \frac{\sin n\alpha}{n^2}] - \frac{1}{n}\cos nx \mid_\alpha^{\pi-\alpha} + \frac{1}{\alpha}\int_\alpha^0 \bar{x}\sin n(\pi - \bar{x})d\bar{x}\Big\} =$$

$$= \frac{2c}{\pi}\Big\{\frac{\sin n\alpha}{\alpha n^2} - \frac{\cos n(\pi-\alpha)}{n} - \frac{1}{\alpha}[\frac{-\alpha\cos n(\alpha-\pi)}{n} + \frac{\sin n(\alpha - \pi)}{n^2}]\Big\} =$$

$$= \frac{2c}{\alpha\pi n^2}\{\sin n\alpha + \sin n(\pi-\alpha)\} = \begin{cases} 0, \text{ if } n \text{ is even} \\ \frac{4c}{\alpha\pi n^2}\sin n\alpha, \text{ if } n \text{ is odd} \end{cases}$$

and $\bar{x} = x - \pi$.
Final answer. $f(x) = \frac{4c}{\pi\alpha}(\sin\alpha\sin x + \frac{\sin 3\alpha}{3^2}\sin 3x + \frac{\sin 5\alpha}{5^2}\sin 5x + \dots)$.

b) With the termwise integration of $x = 2\sum_{n=1}^\infty(-1)^{n+1}\frac{\sin nx}{n}$ *find the Fourier series for* x^2 *and* x^3*.*

One has

$$x^2 = 2\int_0^x xdx = 4\sum_1^\infty(-1)^{n+1}\int_0^x \frac{\sin nx}{n}dx = 4\sum_1^\infty(-1)^n\frac{\cos nx}{n^2}\mid_0^x = 4\sum_1^\infty(-1)^n\frac{\cos nx}{n^2} + 4\sum_1^\infty(-1)^{n+1}\frac{1}{n^2}.$$

The series $4\sum_1^\infty(-1)^{n+1}\frac{1}{n^2}$ equals $\frac{a_0}{2}$, where $a_0 = \frac{2}{\pi}\int_0^\pi x^2dx = \frac{2x^3}{\pi 3}\mid_0^\pi = 2\frac{\pi^2}{3}$.
Therefore, $\frac{a_0}{2} = \frac{\pi^2}{3}$.

$x^3 = 3\int_0^x x^2dx = 3\{\int_0^x \frac{\pi^2}{3}dx + 4\sum_1^\infty(-1)^n\int_0^x \frac{\cos nx}{n^2}dx\} = \pi^2 x + 12\sum_1^\infty(-1)^n\frac{\sin nx}{n^3} = 2\pi^2\sum_1^\infty\frac{(-1)^{n+1}\sin nx}{n} + 12\sum_1^\infty(-1)^n\frac{\sin nx}{n^3} = \sum_1^\infty(-1)^n\sin nx(\frac{12}{n^3} - \frac{2\pi^2}{n}) = \frac{2}{n}\sum_1^\infty(-1)^n\sin nx(\frac{6}{n^2} - \pi^2)$.

2. Solve the boundary-value problem

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(x,0) = u_x(0,t) = u_x(l,t) + hu(l,t) = 0, \quad u_t(x,0) = 1, h > 0 \end{cases}$$

With the separation of variables one gets

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

or A) $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(l) + hX(l) = 0, \end{cases}$ B) $\begin{cases} T'' + \lambda a^2 T = 0 \\ T(0) = 0 \end{cases}$.

The eigenvalues of A) are the solutions of $-\sqrt{\lambda_n}\sin\sqrt{\lambda_n}l + h\cos\sqrt{\lambda_n}l = 0$ or $\sqrt{\lambda_n}\tan\sqrt{\lambda_n}l = h \Rightarrow \tan\sqrt{\lambda_n}l = \frac{h}{\sqrt{\lambda_n}}$.

The eigenfunctions are $X_n = \cos\sqrt{\lambda_n}x$. The corresponding solutions to B) are $T_n = \sin a\sqrt{\lambda_n}t$.

We are looking for a solution in the form $u(x,t) = \sum_1^\infty a_n \cos\alpha_n x \sin a\alpha_n t$, where $\alpha_n = \sqrt{\lambda_n}$.

The condition $u_t(x,0) = 1$ give us $\sum_1^\infty a a_n \alpha_n \cos\alpha_n x = 1$.

We know that the functions $\cos\alpha_n x$ are orthogonal on $[0,l]$ and $\int_0^l \cos^2\alpha_k x dx = \frac{(hl+\sin^2\alpha_k l)}{2h}$, see the textbook.

Thus, $a\alpha_n a_n \int_0^l \cos^2\alpha_n x dx = \int_0^l \cos\alpha_n x dx = \frac{\sin\alpha_n l}{\alpha_n}$, or $a_n = \frac{2h\sin\alpha_n l}{\alpha_n^2 a(hl+\sin^2\alpha_n l)}$.

According to the identity $\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta}$ one gets $\sin^2\alpha_n l = \frac{h^2}{h^2+\alpha_n^2}$.

Because of that we can express a_n as

$$a_n = \frac{2h}{a\alpha_n^2}\frac{\sqrt{\frac{h^2}{h^2+\alpha_n^2}}}{(hl+\frac{h^2}{h^2+\alpha_n^2})} = \frac{2h}{a\alpha_n^2}\frac{\sqrt{h^2+\alpha_n^2}}{l(h^2+\alpha_n^2)+h}.$$

3. Solve the boundary-value problem

$$\begin{cases} u_{xx}+u_{yy}=0; \;\; 0<x<1, \; 0<y<1 \\ u_x(0,y)=u_x(1,y)=0, \;\; u(x,0)=A, \;\; u(x,1)=Bx. \end{cases}$$

Due to the superposition principle we are looking for a solution as the sum of the harmonic function which satisfies I) $u_x(0,y) = u_x(1,y) = u(x,0) = 0$ och $u(x,1) = Bx$ and the harmonic function satisfying II) $u_x(0,y) = u_x(1,y) = u(x,1) = 0$ and $u(x,0) = A$.

Separation of variables give us

1) $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(1) = 0, \end{cases}$

2) $\begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = 0 \end{cases}$.

Problem I) has the following solutions: $\lambda = 0, X_0 = 1$ and $\lambda_k^2 = \pi^2(2k+1)^2$; with $X_k = \cos\pi(2k+1)x$. The corresponding solutions to problem 2) are $Y_k = \sin h\pi(2k+1)y$ and $Y_0 = y$.

Thus we are looking for $u(x,y)$ in the form

$$u(x,y) = \frac{a_0 y}{2} + \sum_{k=1}^\infty a_k \cos\pi(2k-1)x \sin h\pi(2k-1)y.$$

The last condition gives us

$$u(x,1) = \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos\pi(2k-1)x \sin h\pi(2k-1) = Bx.$$

This means that $a_0 = 2\int_0^1 Bxdx = B$ and

$$a_k \sin h\pi(2k-1) = 2\int_0^1 Bx\cos\pi(2k-1)dx = 2B\left\{x\frac{\sin\pi(2k-1)x}{\pi(2k-1)}\,|_0^1 - \int_0^1 \frac{\sin\pi(2k-1)x}{\pi(2k-1)dx}\right\} = = \frac{2B\cos\pi(2k -}{\pi^2(2k-1}$$

Finally, $a_k = \frac{-4B}{\pi^2(2k-1)^2\sin h(2k-1)\pi}$.

Problem II has an obvious solution $A - Ay$.

Answer. $u(x,t) = A + y(\frac{B}{2} - A) - \sum_{k=1}^\infty \frac{4B}{\pi^2(2k-1)^2\sin h(2k-1)\pi}\cos\pi(2k-1)x\sin h\pi(2k-1)y$.

4. a) Find the eigenvalues and the eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} (x^2y')' + \lambda X = 0; \quad \text{on } [1,4] \\ y(1) = y(4) = 0. \end{cases}$$

Using the variable change $x = e^s$ we obtain

$$\frac{d^2y}{ds^2} + \frac{dy}{ds} + \lambda s = 0; \quad y(0) = y(\ln 4) = 0.$$

A general solution to the new problem is:

$$y = e^{\frac{-s}{2}}(a \sin s\sqrt{\lambda - \frac{1}{4}} + b \cos s\sqrt{\lambda - \frac{1}{4}})$$

and since $y(0) = y(\ln 4) = 0$ we get $b = 0$ and $\ln 4\sqrt{\lambda_k - \frac{1}{4}} = \pi k$.

The eigenvalues are $\alpha_k = \sqrt{\lambda_k - \frac{1}{4}} = \frac{\pi k}{\ln 4}$ and the eigenfunctions are $y_k = \frac{1}{\sqrt{x}} \frac{\sin \pi k \ln x}{\ln 4}$. *b) Is the operator* $Ly = (1 - x^2)y'' - 2xy' + n(n+1)y$ *self-adjoint?*

$L^*y = ((1-x^2)y)'' - (-2xy)' + n(n+1)y = ((1-x^2)y' - 2xy)' + 2(xy' + y) + n(n+1)y = (1 - x^2)y'' - 2xy' + n(n+1)y.$

Answer: Ly is self-adjoint.

5. a) Represent the function $f(x) = \begin{cases} 1 - x, & \textit{där } 0 < x < 1 \\ 0, & \textit{där } 1 < x < \infty. \end{cases}$
as a Fourier sine integral.

We are looking for a representation in the form: $f(x) = \int_0^\infty B(\alpha) \sin \alpha x dx$ where $B(\alpha) = \frac{2}{\pi} \int_0^\infty f(x) \sin \alpha x dx = \frac{2}{\pi} \int_0^1 (1 - x) \sin \alpha x dx = \frac{2}{\pi}\{\frac{-\cos \alpha x}{\alpha} \mid_0^1 + \frac{x \cos \alpha x}{\alpha} \mid_0^1 - \frac{\sin \alpha x}{\alpha}\} = \frac{2(\alpha - \sin \alpha)}{\pi \alpha^2}$.

Answer. $f(x) = \frac{2}{\pi} \int_0^\infty (\frac{\alpha - \sin \alpha}{\alpha^2}) \sin \alpha x d\alpha$.

b) Calculate $\int_0^\infty \frac{(\alpha - \sin \alpha) \sin \frac{\alpha}{2} d\alpha}{\alpha^2}$.

If $x = \frac{1}{2}$ then $\frac{1}{2} = f(\frac{1}{2}) = \frac{2}{\pi} \int_0^\infty \frac{(\alpha - \sin \alpha)}{\alpha^2} \sin \alpha x d\alpha$ or $\frac{\pi}{4} = \int_0^\infty \frac{(\alpha - \sin \alpha)}{\alpha^2} \sin \alpha x d\alpha$.

6. A function f is defined on the whole x-axis and has a derivative there. Show that under the conditions that f and f' are absolutely integrable one has that

$$\lim_{\lambda \to \infty} \int_{-\infty}^{+\infty} f(x) \sin \lambda x dx = 0.$$

It suffices to show that for any $\epsilon > 0$ there exists R such that for each $\lambda > R \Rightarrow \mid \int_{-\infty}^\infty f(\lambda) \sin \lambda x dx \mid < \epsilon$.

The condition that $f$is absolutely integrable implies $\Rightarrow \exists N$ such that $\int_{-\infty}^{-N} \mid f(x) \mid dx + \int_N^\infty \mid f(x) \mid dx < \epsilon/2$.

Since f is differentiable $\Rightarrow f$ is continuous $\Rightarrow f$ is bounded ($\mid f \mid \leq M$) on the interval $[-N, N]$.

$$\mid \int_{-N}^{N} f(x) \sin \lambda x dx \mid = \mid [\frac{-\cos \lambda x}{\lambda} f(x)]_{-N}^N + \int_{-N}^N \frac{\cos \lambda x}{\lambda} f'(x) dx \mid \leq 2M/\lambda + \frac{1}{\lambda} \int_{-\infty}^\infty \mid f'(x) \mid dx < \epsilon/2$$

om $\lambda > R = \frac{(4M + 2\int_{-\infty}^\infty |f'(x)| dx)}{\epsilon}$.

Thus $\lambda > R \Rightarrow \mid \int_{-\infty}^\infty f(x) \sin \lambda x dx \mid \leq \int_{-\infty}^{-N} + \int_{-N}^N + \int_N^\infty < \epsilon/2 + \epsilon/2 = \epsilon$. □