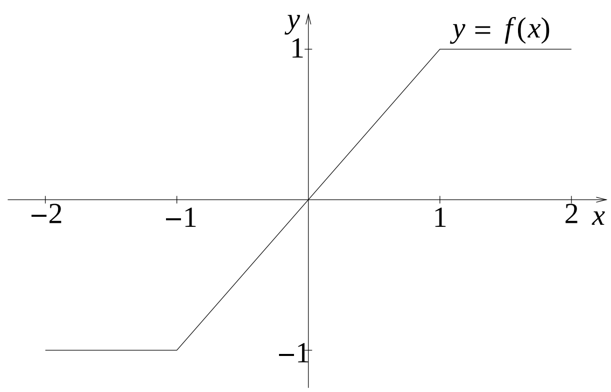
MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sara Maad Sasane

Tentamensskrivning i
Linjär analys
7,5 hp
2009-12-16

Only writing materials are allowed. No calculators are allowed. The problems are not ordered according to difficulty. Results from a subproblem may be used in the following subproblems even if the earlier subproblem has not been solved. Motivate your solutions carefully. The limits for the grades are as follows: A: 90–100 points, B: 75–89 points, C: 65–74 points, D: 55–64 points, E: 50–54 points, Fx: 40–49 points, F: 0–39 points. For passing at least the grade E is required on the written exam as well as a passing grade on the mandatory home assignment.
Good luck!

1. Let f be the function whose graph is shown in Figure 1. Determine the Fourier series for f on the interval $(-2, 2)$. 10 p



Figur 1: The graph of f

2. a) Determine the cosine series for $f(x) = x^2$ on the interval $0 < x < \pi$. 10 p

   b) Motivate why the series that you determined in the subproblem a) converges pointwise to x^2 on the interval $-\pi \le x \le \pi$. 4 p

   c) Show that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad \text{and that} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
6 p

3. a) Show that
$$e^{-|x|} = \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+\alpha^2} \cos(\alpha x)\, d\alpha, \qquad x \in \mathbb{R}.$$
10 p

   b) Determine the Fourier cosine integral of the function
$$f(x) = \frac{1}{1+x^2},$$
i.e. determine $A(\alpha)$ so that
$$f(x) = \int_0^{\infty} A(\alpha) \cos(\alpha x)\, d\alpha.$$
5 p

PLEASE TURN OVER!

4. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and 2π-periodic and that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Show Wirtinger's inequality, i.e.

$$\int_{-\pi}^{\pi} f(x)^2\,dx \le \int_{-\pi}^{\pi} f'(x)^2\,dx,$$

and that the inequality is an equality if and only if $f(x) = A\cos(x) + B\sin(x)$, where A and B are constants. *Hint: You are allowed to use the fact that if g is a function which is piecewise continuous on $\mathbb{R}$, then the Parseval identity holds, i.e.*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

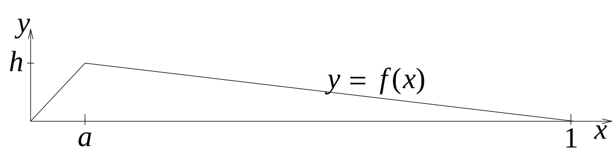
where a_n and b_n are the standard Fourier coefficients of g on the interval $[-\pi, \pi]$. 10 p

5. A string of length 1 is stretched between the points $x = 0$ and $x = 1$ on the x axis. The string is stretched so that it gets the profile $y = f(x)$, where f is the function whose graph is shown in Figure 2 and is then released from this position at the time $t = 0$. We assume that the vertical displacement of the string y from the x-axis at the point x and time t is determined by the wave equation

$$y_{tt} = c^2 y_{xx}, \qquad 0 < x < 1,\ t > 0,$$

where c is a positive constant, together with the boundary and initial conditions

$$y(0,t) = y(1,t) = 0, \qquad t > 0,$$
$$y(x,0) = f(x), \quad y_t(x,0) = 0, \qquad 0 < x < 1.$$



Figur 2: The function f describes the initial values of y.

a) Determine the sine series of f on the interval $[0, 1]$, and show that the series converges uniformly to f on this interval. 5 p

b) Use separation of variables to show that

$$y(x,t) = \frac{2h}{\pi^2 a(1-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi a)\cos(n\pi ct)\sin(n\pi x)$$

is a formal solution of this boundary value problem. 10 p

c) Use trigonometric formulas and a theorem about uniform convergence of Fourier series to show that the formal solution that was obtained in subproblem a) converges uniformly to

$$y(x,t) = \frac{1}{2}(F(x-ct) + F(x+ct)),$$

where F is the odd periodic extension of f of period 2. 5 p

PLEASE TURN OVER!

d) Show that y from subproblem c) solves the wave equation for every $(x, t) \in (0, 1) \times (0, \infty)$ such that $(x \pm a \pm ct)/2 \notin \mathbb{Z}$ (for all four combinations of $\pm$), and that the given initial and boundary conditions are satisfied. 4 p

e) Let $c = 1$ and $h = a = 0.1$. Draw snapshot pictures of the solution $y(x, t)$ for $x \in [0, 1]$ and $t = 0$, $t = 0.2$, $t = 0.4$. 6 p

6. a) Let $u(r, t)$ denote the temperature in a spherical shell $1 \leq r \leq b$, where r is the radial coordinate and the shell in the beginning has the temperature $u(r, 0) = f(r)$. Both the inner and the outer surfaces $r = 1$ and $r = b$ are kept at the constant temperature zero. The temperature in the shell is following the heat equation, which in spherical coordinates without ϕ- and θ-depencence can be written

$$u_t = \frac{k}{r}(ru)_{rr} \qquad (1 < r < b,\ t > 0),$$

where k is a positive constant. Show formally that the temperature in the shell when $t > 0$ is

$$u(r, t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{r-1}{b-1}\right)\exp\left(-\frac{n^2\pi^2 k}{(b-1)^2}t\right),$$

where

$$B_n = \frac{2}{b-1}\int_1^b r\,f(r)\sin\left(n\pi\frac{r-1}{b-1}\right)dr.$$

Hint: The change of variables $X(r) = rR(r)$ can be of help vara for solving one of the equations that occur at the separation of variables. 10 p

b) What is the temperature development in the shell if the initial temperature is

$$f(r) = \frac{1}{r}\sin\left(\pi\frac{r-1}{b-1}\right), \qquad (1 < r < b)?$$

5 p

The exams will be returned on Monday 21 December, at 12.45–13.00 in room 311, house 6; or at a later date in Reine Elfsö's office room 208, house 6.