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Avd. Matematik
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Lösningar till tentamensskrivning i
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7.5 hp
2009-12-16

1. Since f is an odd function, the coefficients a_n in the Fourier series are all zero. Also, for every $n \geq 1$, $f(x)\sin(n\pi x/2)$ is an even function, and so

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)dx = \int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx.$$

We then calculate

$$\begin{aligned} b_n &= \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx + \int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2}{n\pi}\left[x\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx - \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 \\ &= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) \\ &= \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}(-1)^{n+1}, \end{aligned}$$

and so

$$f(x) \sim \frac{2}{n\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + (-1)^{n+1}\right)\sin\left(\frac{n\pi}{2}x\right), \qquad -2 < x < 2.$$

2. a) We calculate

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}$$

and for $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{n\pi}\left[x^2\sin(nx)\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x\sin(nx)\,dx \\ &= \frac{4}{n^2\pi}\left[x\cos(nx)\right]_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{4}{n^2}(-1)^n. \end{aligned}$$

Hence the cosine series of x^2 on $(0, \pi)$ is

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \qquad 0 < x < \pi.$$

b) Since x^2 is an even function and the functions $\cos(nx)$ are all even, the Fourier series on $-\pi < x < \pi$ is given by

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \qquad -\pi < x < \pi.$$

Since $f(x) = x^2$ is even it satisfies $f(\pi) = f(-\pi)$, and so the periodic extension F of f is continuous on $\mathbb{R}$. In particular $F(x) = (F(x_+) + F(x_-))$ for every $x \in \mathbb{R}$. Moreover $f'(x)$ is piecewise continuous, and so f is piecewise smooth on $(-\pi, \pi)$. A Corollary in the book (Section 12) now says that the Fourier series of x^2 above converges to $F(x)$ for every $x \in \mathbb{R}$. In particular this series converges to x^2 on $[-\pi, \pi]$.

c) By subproblem b), we can let $x = 0$ and $x = \pi$, respectively, and we obtain both identities.

3. a) Note that $f(x) = e^{-|x|}$ is absolutely integrable and continuous and bounded on $\mathbb{R}$ (and hence piecewise continuous on every bounded interval of $\mathbb{R}$). Moreover, $f_R'(x)$ and $f_L'(x)$ exist everywhere on $\mathbb{R}$. Since

$f(x)$ is an even function, its Fourier integral coincides with its Fourier cosine integral, and so the Fourier integral representation of $e^{-|x|}$ is given by

$$e^{-|x|} = \int_0^\infty A(\alpha)\cos(\alpha x)\,d\alpha, \qquad (x \in \mathbb{R}),$$

where

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty e^{-|s|}\cos(\alpha s)\,ds.$$

We compute $A(\alpha)$ and note that $e^{-|s|} = e^{-s}$ on $\mathbb{R}_+$. By integration by parts we have

$$\begin{aligned} A(\alpha) &= \frac{2}{\pi}\int_0^\infty e^{-s}\cos(\alpha(s))\,ds = \frac{2}{\pi}\left[-e^{-s}\cos(\alpha s)\right]_0^\infty - \frac{2}{\pi}\alpha\int_0^\infty e^{-s}\sin(\alpha s)\,ds \\ &= \frac{2}{\pi} + \frac{2}{\pi}\alpha\left[e^{-s}\sin(\alpha s)\right]_0^\infty - \frac{2}{\pi}\alpha^2\int_0^\infty e^{-s}\cos(\alpha s)\,ds \\ &= \frac{2}{\pi} - \alpha^2 A(\alpha), \end{aligned}$$

and so

$$A(\alpha) = \frac{2}{\pi}\frac{1}{1+\alpha^2}.$$

b) Arguing as in subproblem a), the function f has a Fourier integral representation given by

$$f(x) = \int_0^\infty A(\alpha)\cos(\alpha x)\,dx,$$

where

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2}\cos(\alpha s)\,ds.$$

Noting that the integral appearing in the formula for $A(\alpha)$ also appears in subproblem a) with s replaced by α and α replaced by x, we see immediately that

$$A(\alpha) = e^{-|\alpha|} = e^{-\alpha}$$

for $\alpha > 0$, and so we can write

$$\frac{1}{1+x^2} = \int_0^\infty e^{-\alpha}\cos(\alpha x)\,d\alpha.$$

4. We note that the conditions for Parseval's formula are valid for both f and f'. We denote by a_n and b_n the Fourier coefficients for f and by α_n and β_n the Fourier coefficients for f', i.e.

$$a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad \alpha_n = \int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx, \qquad n = 0, 1, 2, \ldots$$

and

$$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \qquad \beta_n = \int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx, \qquad n = 1, 2, 3, \ldots$$

By the assumption that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

it follows that $a_0 = 0$. Moreover,

$$\alpha_0 = \int_{-\pi}^{\pi} f'(x)\,dx = f(\pi) - f(-\pi) = 0.$$

For $n \geq 1$ we use integration by parts and see that

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx = \frac{1}{\pi}\left[f(x)\cos(nx)\right]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left((-1)^n(f(\pi)-f(-\pi))\right) + nb_n = nb_n,$$

$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx = \frac{1}{\pi}\left[f(x)\sin(nx)\right]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = -na_n.$$

Combining this with Parseval's formula, and noticing that $1/n^2 \leq 1$ for all $n \geq 1$, we see that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\frac{\alpha_n^2+\beta_n^2}{n^2} \leq \sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)^2\,dx.$$

After multiplying both sides by π, the desired inequality is obtained. We also note that for equality to hold, we need $a_n^2 + b_n^2 = 0$ for every $n \geq 2$. For this to happen, we must have $f(x) = a_1\cos(x) + b_1\sin(x)$, as required. Conversely, it is clear that the inequality is an equality if $f(x) = A\sin(x) + B\cos(x)$.

5. a) The function f is given by the formula

$$f(x) = \begin{cases} \frac{h}{a}x & \text{for } 0 \leq x \leq a, \\ \frac{h}{1-a}(1-x) & \text{for } a \leq x \leq 1. \end{cases}$$

Its Fourier sine series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

where

$$\begin{aligned}
b_n &= 2\int_0^{\pi} f(x)\sin(n\pi x)\,dx = 2\left(\int_0^a \frac{h}{a}x\sin(n\pi x)\,dx + \int_a^1 \frac{h}{1-a}(1-x)\sin(n\pi x)\,dx\right) \\
&= -\frac{2h}{n\pi a}\left[x\cos(n\pi x)\right]_0^a + \frac{2h}{n\pi a}\int_0^a \cos(n\pi x)\,dx - \frac{2h}{n\pi(1-a)}\left[(1-x)\cos(n\pi x)\right]_a^1 - \frac{2h}{n\pi(1-a)}\int_a^1 \cos(n\pi x)\,dx \\
&= -\frac{2h}{n\pi}\cos(n\pi a) + \frac{2h}{n^2\pi^2 a}\sin(n\pi a) + \frac{2h}{n\pi}\cos(n\pi a) + \frac{2h}{n^2\pi^2(1-a)}\sin(n\pi a) \\
&= \frac{2h}{n^2\pi^2 a(1-a)}\sin(n\pi a)
\end{aligned}$$

for $n = 1, 2, 3, \ldots$

b) We look for solutions of the form $y(x,t) = X(x)T(t)$. The wave equation then reduces to

$$X(x)T''(t) = c^2X''(x)T(t), \qquad (0 < x < 1, t > 0).$$

For nontrivial solutions X and T we then have

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. Hence

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T''(t) + \lambda cT(t) &= 0. \end{aligned}$$

The homogeneous boundary conditions for X and T are

$$X(0) = X(1) = 0, \qquad T'(0) = 0.$$

If $\lambda > 0$, it follows that $X(x) = B\sin(n\pi x)$, and we choose $B = 1$ for convenience. The corresponding λ is $n^2\pi^2$. Then $T(t) = C\cos(n\pi ct)$, and again we let $C = 1$.

It is easy to check that there are no nontrivial solutions for $\lambda \leq 0$. Hence a formal solution is given by

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct)\sin(n\pi x).$$

We will use the inhomogeneous boundary condition to determine the constants B_n. We have

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = f(x),$$

from which we see that B_n has to be the Fourier sine coefficients for f on the interval $(0,1)$. From the result in subproblem a) we see that

$$B_n = b_n = \frac{2h}{n^2\pi^2 a(1-a)}\sin(n\pi a)$$

After inserting this B_n into the formula for the formal solution, we obtain

$$y(x,t) = \frac{2h}{\pi^2 a(1-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi a)\cos(n\pi ct)\sin(n\pi x)$$

c) Using that

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B)+\sin(A-B)),$$

we get

$$y(x,t) = \frac{h}{\pi^2 a(1-a)}\sum_{n=1}^{\infty}\frac{\sin(n\pi a)}{n^2}\left(\sin(n\pi(x+ct))+\sin(n\pi(x-ct))\right),$$

and so it suffices to show that

$$F(x) = \frac{2h}{\pi^2 a(1-a)}\sum_{n=1}^{\infty}\frac{\sin(n\pi a)}{n^2}\sin(n\pi x)$$

converges uniformly on $\mathbb{R}$ and that F is the odd 2-periodic extension of f. The uniform convergence can be shown by using Weierstrass' M-test. Indeed, the absolute value of the terms in the series are dominated by a constant multiplied by $1/n^2$, and the series formed by these terms converges. (Alternatively uniform convergence can be shown by arguing that F is continuous and 2-periodic on $\mathbb{R}$ and piecewise smooth on $(-1,1)$. A theorem in the book (p. 48) shows that the series converges uniformly).
From subproblem a) it is clear that $F(x) = f(x)$ for $0 \leq x \leq \pi$. Since F is odd and 2-periodic it follows that F is the 2-periodic extension of f.

d) Let $y(x,t)$ be as in subproblem c), where F is the odd periodic extension of f. F is infinitely differentiable at all $x \in \mathbb{R}$, except at the points $a+2n$ and $-a+2n$, where $n \in \mathbb{Z}$. By the chain rule it follows that

$$y_{tt}(x,t) = \frac{c^2}{2}(F''(x-ct)+F''(x+ct)),$$
$$y_{xx}(x,t) = \frac{1}{2}(F''(x-ct)+F''(x+ct)),$$

for all x and t such that $F''(x-ct)$ and $F''(x+ct)$ exist, i.e. for all $x \in (0,1)$ and $t > 0$ except those satisfying $(x+ct-a)/2 \in \mathbb{Z}$, $(x+ct+a)/2 \in \mathbb{Z}$, $(x-ct-a)/2 \in \mathbb{Z}$, or $(x-ct+a)/2 \in \mathbb{Z}$. It is clear that y satisfies the wave equation for every x except for the points in the above mentioned set. To verify that y also satisfies the boundary conditions, we verify

$$y(0,t) = \frac{1}{2}(F(-ct)+F(ct)) = 0,$$
$$y(1,t) = \frac{1}{2}(F(1-ct)+F(1+ct)) = 0,$$

where the first equality holds since F is odd, and the second holds since F is 2-periodic and odd. Furthermore,

$$y(x,0) = \frac{1}{2}(F(x) + F(x)) = f(x), \qquad x \in [0,1],$$
$$y_t(x,0) = \frac{c}{2}(-F'(x) + F'(x)) = 0, \qquad x \in [0,1],$$

as required.

e) For $t = 0$ the graph of $y(x,t)$ is just the graph of f (where $a = h = 0.1$). We also know that $y(0,t) = y(1,t) = 0$ for every $t > 0$. For $t = 0.2$, we have $y(x,0.2) = (F(x+0.2) + F(x-0.2))/2$. It is clear that the graph of $F(x-0.2)$ has a corner for x in the interval $[0,1]$ when $x - 0.2 = 0.1$ (using the periodicity of F), that is when $x = 0.3$, and when $x - 0.2 = -0.1$, that is when $x = 0.1$. The graph of $F(x+0.2)$ does not have any jumps in the derivative in the interval $[0,1]$. The value of $y(x,0.2)$ at these two corner points are

$$y(0.1,0.2) = (F(0.3) + F(-0.1))/2 = (0.7/9 - 0.1)/2 = -0.1/9 \approx -0.01,$$
$$y(0.3,0.2) = (F(0.5) + F(0.1))/2 = (0.5/9 + 0.1)/2 = 0.7/9 \approx 0.78.$$

The graph is constructed by joining these known points with straight lines.
Similarly, when $t = 0.4$, we get corner points for $F(x-0.4)$ when $x - 0.4 = 0.1$ or $x - 0.4 = -0.1$, that is when $x = 0.5$ or $x = 0.3$. The values of $y(x,0.4)$ for these points are

$$y(0.3,0.4) = (F(0.7) + F(-0.1))/2 = (0.3/9 - 0.1)/2 = -1/30 \approx -0.03,$$
$$y(0.5,0.2) = (F(0.9) + F(0.1))/2 = (0.1/9 + 0.1)/2 = 1/18 \approx 0.06.$$

Again, the graph is drawn by joining the four known points with straight lines (figure will be added soon).

6. a) We use separation of variables, and look for solutions of the form $u(r,t) = R(r)T(t)$. For such functions, the equation becomes

$$R(r)T'(t) = \frac{k}{r}(rR(r))''T(t).$$

If u is a nontrivial solution, we can divide by $kR(r)T(t)$ and obtain

$$\frac{T'(t)}{kT(t)} = \frac{(rR(r))''}{rR(r)}.$$

The left hand side and the right hand side are independent of r and t, respectively, and since they are equal they depend neither on r nor t, and so they are constant, say $-\lambda$. By rearranging and including the homogeneous boundary conditions, we end up with the following system of ODEs:

$$(rR(r))'' + \lambda rR(r) = 0,$$
$$T'(t) + \lambda kT(t) = 0.$$

The change of variables $X(r) = rR(r)$ transforms the first of these two equations to $X''(x) + \lambda X(x) = 0$. The appropriate boundary condition is $X(1) = bX(b) = 0$, i.e. $X(1) = X(b) = 0$. Assume first that $\lambda = \alpha^2$ for some $\alpha > 0$. If X satisfies the ODE and the first boundary condition, then

$$X(r) = B\sin(\alpha(r-1)).$$

If also the second boundary condition is satisfied, then $\alpha(b-1) = n\pi$, i.e. $\alpha = n\pi/(b-1)$. For convenience, we let $B = 1$ and let

$$R_n(r) = \frac{1}{r}\sin\left(n\pi\frac{r-1}{b-1}\right).$$

It can be checked that there are no nontrivial solutions if $\lambda \leq 0$. If $\lambda = n^2\pi^2/(b-1)^2$, the equation for T has the solutions

$$T_n(t) = C\exp\left(-\frac{n^2\pi^2 k}{(b-1)^2}t\right).$$

As usual, we choose $C = 1$. A formal solution is a generalized linear combination

$$u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{r-1}{b-1}\right)\exp\left(-\frac{n^2\pi^2 k}{(b-1)^2}t\right),$$

where B_n are constants. These constants are determined by the initial condition $u(r,0) = f(r)$. We have

$$u(r,0) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{r-1}{b-1}\right) = f(r) = \frac{1}{r}rf(r).$$

The function $rf(r)$ has a Fourier sine expansion

$$rf(r) = \frac{2}{b-1}\sum_{n=1}^{\infty}\int_1^b sf(s)\sin\left(n\pi\frac{s-1}{b-1}\right)ds\sin\left(n\pi\frac{r-1}{b-1}\right)$$

on the interval $(1,b)$.

b) For this $f(r)$ we have

$$rf(r) = \sin\left(\pi\frac{r-1}{b-1}\right),$$

which is already expanded in a Fourier series (with only one term) on the interval $(1,b)$. By identifying coefficients, we see that $B_1 = 1$ and $B_n = 0$ for $n \geq 2$. From subproblem a) we then see that

$$u(r,t) = \frac{1}{r}\sin\left(\pi\frac{r-1}{b-1}\right)\exp\left(-\frac{\pi^2 k}{(b-1)^2}t\right), \qquad (1 < r < b,\ t > 0).$$