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Avd. Matematik
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Lösningar till tentamensskrivning i
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1. Since f is an odd function, the coefficients a_n in the Fourier series are all zero. Also, for every $n \geq 1$, $f(x)\sin(n\pi x/2)$ is an even function, and so

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)dx = \int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx.$$

We then calculate

$$\begin{aligned} b_n &= \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx + \int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2}{n\pi}\left[x\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx - \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 \\ &= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) \\ &= \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}(-1)^{n+1}, \end{aligned}$$

and so

$$f(x) \sim \frac{2}{n\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + (-1)^{n+1}\right)\sin\left(\frac{n\pi}{2}x\right), \qquad -2 < x < 2.$$

2. a) We calculate

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}$$

and for $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{n\pi}\left[x^2\sin(nx)\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x\sin(nx)\,dx \\ &= \frac{4}{n^2\pi}\left[x\cos(nx)\right]_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{4}{n^2}(-1)^n. \end{aligned}$$

Hence the cosine series of x^2 on $(0, \pi)$ is

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \qquad 0 < x < \pi.$$

b) Since x^2 is an even function and the functions $\cos(nx)$ are all even, the Fourier series on $-\pi < x < \pi$ is given by

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \qquad -\pi < x < \pi.$$

Since $f(x) = x^2$ is even it satisfies $f(\pi) = f(-\pi)$, and so the periodic extension F of f is continuous on $\mathbb{R}$. In particular $F(x) = (F(x_+) + F(x_-))$ for every $x \in \mathbb{R}$. Moreover $f'(x)$ is piecewise continuous, and so f is piecewise smooth on $(-\pi, \pi)$. A Corollary in the book (Section 12) now says that the Fourier series of x^2 above converges to $F(x)$ for every $x \in \mathbb{R}$. In particular this series converges to x^2 on $[-\pi, \pi]$.

c) By subproblem b), we can let $x = 0$ and $x = \pi$, respectively, and we obtain both identities.

3. a) Note that $f(x) = e^{-|x|}$ is absolutely integrable and continuous and bounded on $\mathbb{R}$ (and hence piecewise continuous on every bounded interval of $\mathbb{R}$). Moreover, $f_R'(x)$ and $f_L'(x)$ exist everywhere on $\mathbb{R}$. Since

$f(x)$ is an even function, its Fourier integral coincides with its Fourier cosine integral, and so the Fourier integral representation of $e^{-|x|}$ is given by

$$e^{-|x|} = \int_0^\infty A(\alpha)\cos(\alpha x)\,d\alpha, \qquad (x \in \mathbb{R}),$$

where

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty e^{-|s|}\cos(\alpha s)\,ds.$$

We compute $A(\alpha)$ and note that $e^{-|s|} = e^{-s}$ on $\mathbb{R}_+$. By integration by parts we have

$$\begin{aligned} A(\alpha) &= \frac{2}{\pi}\int_0^\infty e^{-s}\cos(\alpha(s))\,ds = \frac{2}{\pi}\left[-e^{-s}\cos(\alpha s)\right]_0^\infty - \frac{2}{\pi}\alpha\int_0^\infty e^{-s}\sin(\alpha s)\,ds \\ &= \frac{2}{\pi} + \frac{2}{\pi}\alpha\left[e^{-s}\sin(\alpha s)\right]_0^\infty - \frac{2}{\pi}\alpha^2\int_0^\infty e^{-s}\cos(\alpha s)\,ds \\ &= \frac{2}{\pi} - \alpha^2 A(\alpha), \end{aligned}$$

and so

$$A(\alpha) = \frac{2}{\pi}\frac{1}{1+\alpha^2}.$$

b) Arguing as in subproblem a), the function f has a Fourier integral representation given by

$$f(x) = \int_0^\infty A(\alpha)\cos(\alpha x)\,dx,$$

where

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2}\cos(\alpha s)\,ds.$$

Noting that the integral appearing in the formula for $A(\alpha)$ also appears in subproblem a) with s replaced by α and α replaced by x, we see immediately that

$$A(\alpha) = e^{-|\alpha|} = e^{-\alpha}$$

for $\alpha > 0$, and so we can write

$$\frac{1}{1+x^2} = \int_0^\infty e^{-\alpha}\cos(\alpha x)\,d\alpha.$$

4. We note that the conditions for Parseval's formula are valid for both f and f'. We denote by a_n and b_n the Fourier coefficients for f and by α_n and β_n the Fourier coefficients for f', i.e.

$$a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad \alpha_n = \int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx, \qquad n = 0, 1, 2, \ldots$$

and

$$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \qquad \beta_n = \int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx, \qquad n = 1, 2, 3, \ldots$$

By the assumption that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

it follows that $a_0 = 0$. Moreover,

$$\alpha_0 = \int_{-\pi}^{\pi} f'(x)\,dx = f(\pi) - f(-\pi) = 0.$$

For $n \geq 1$ we use integration by parts and see that

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx = \frac{1}{\pi}\left[f(x)\cos(nx)\right]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left((-1)^n(f(\pi) - f(-\pi))\right) + nb_n = nb_n,$$

$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx = \frac{1}{\pi}\left[f(x)\sin(nx)\right]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = -na_n.$$

Combining this with Parseval's formula, and noticing that $1/n^2 \leq 1$ for all $n \geq 1$, we see that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\frac{\alpha_n^2 + \beta_n^2}{n^2} \leq \sum_{n=1}^{\infty}(\alpha_n^2 + \beta_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)^2\,dx.$$

After multiplying both sides by π, the desired inequality is obtained. We also note that for equality to hold, we need $a_n^2 + b_n^2 = 0$ for every $n \geq 2$. For this to happen, we must have $f(x) = a_1\cos(x) + b_1\sin(x)$, as required. Conversely, it is clear that the inequality is an equality if $f(x) = A\sin(x) + B\cos(x)$.

5. a) The function f is given by the formula

$$f(x) = \begin{cases} \frac{h}{a}x & \text{for } 0 \leq x \leq a, \\ \frac{h}{1-a}(1-x) & \text{for } a \leq x \leq 1. \end{cases}$$

Its Fourier sine series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

where

$$\begin{aligned} b_n &= 2\int_0^{\pi} f(x)\sin(n\pi x)\,dx = 2\left(\int_0^a \frac{h}{a}x\sin(n\pi x)\,dx + \int_a^1 \frac{h}{1-a}(1-x)\sin(n\pi x)\,dx\right) \\ &= -\frac{2h}{n\pi a}\left[x\cos(n\pi x)\right]_0^a + \frac{2h}{n\pi a}\int_0^a \cos(n\pi x)\,dx - \frac{2h}{n\pi(1-a)}\left[(1-x)\cos(n\pi x)\right]_a^1 - \frac{2h}{n\pi(1-a)}\int_a^1 \cos(n\pi x)\,dx \\ &= -\frac{2h}{n\pi}\cos(n\pi a) + \frac{2h}{n^2\pi^2 a}\sin(n\pi a) + \frac{2h}{n\pi}\cos(n\pi a) + \frac{2h}{n^2\pi^2(1-a)}\sin(n\pi a) \\ &= \frac{2h}{n^2\pi^2 a(1-a)}\sin(n\pi a) \end{aligned}$$

for $n = 1, 2, 3, \ldots$

b) We look for solutions of the form $y(x,t) = X(x)T(t)$. The wave equation then reduces to

$$X(x)T''(t) = c^2X''(x)T(t), \qquad (0 < x < 1, t > 0).$$

For nontrivial solutions X and T we then have

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. Hence

$$X''(x) + \lambda X(x) = 0,$$
$$T''(t) + \lambda cT(t) = 0.$$

The homogeneous boundary conditions for X and T are

$$X(0) = X(1) = 0, \qquad T'(0) = 0.$$

If $\lambda > 0$, it follows that $X(x) = B\sin(n\pi x)$, and we choose $B = 1$ for convenience. The corresponding λ is $n^2\pi^2$. Then $T(t) = C\cos(n\pi ct)$, and again we let $C = 1$.

It is easy to check that there are no nontrivial solutions for $\lambda \le 0$. Hence a formal solution is given by

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct)\sin(n\pi x).$$

We will use the inhomogeneous boundary condition to determine the constants B_n. We have

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = f(x),$$

from which we see that B_n has to be the Fourier sine coefficients for f on the interval $(0,1)$. From the result in subproblem a) we see that

$$B_n = b_n = \frac{2h}{n^2\pi^2 a(1-a)}\sin(n\pi a)$$

After inserting this B_n into the formula for the formal solution, we obtain

$$y(x,t) = \frac{2h}{\pi^2 a(1-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi a)\cos(n\pi ct)\sin(n\pi x)$$

c) Using that

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B)+\sin(A-B)),$$

we get

$$y(x,t) = \frac{h}{\pi^2 a(1-a)}\sum_{n=1}^{\infty}\frac{\sin(n\pi a)}{n^2}\left(\sin(n\pi(x+ct)) + \sin(n\pi(x-ct)\right),$$

and so it suffices to show that

$$F(x) = \frac{2h}{\pi^2 a(1-a)}\sum_{n=1}^{\infty}\frac{\sin(n\pi a)}{n^2}\sin(n\pi x)$$

converges uniformly on $\mathbb{R}$ and that F is the odd 2-periodic extension of f. The uniform convergence can be shown by using Weierstrass' M-test. Indeed, the absolute value of the terms in the series are dominated by a constant multiplied by $1/n^2$, and the series formed by these terms converges. (Alternatively uniform convergence can be shown by arguing that F is continuous and 2-periodic on $\mathbb{R}$ and piecewise smooth on $(-1,1)$. A theorem in the book (p. 48) shows that the series converges uniformly).

From subproblem a) it is clear that $F(x) = f(x)$ for $0 \le x \le \pi$. Since F is odd and 2-periodic it follows that F is the 2-periodic extension of f.

d) Let $y(x,t)$ be as in subproblem c), where F is the odd periodic extension of f. F is infinitely differentiable at all $x \in \mathbb{R}$, except at the points $a+2n$ and $-a+2n$, where $n \in \mathbb{Z}$. By the chain rule it follows that

$$y_{tt}(x,t) = \frac{c^2}{2}(F''(x-ct) + F''(x+ct)),$$
$$y_{xx}(x,t) = \frac{1}{2}(F''(x-ct) + F''(x+ct)),$$

for all x and t such that $F''(x-ct)$ and $F''(x+ct)$ exist, i.e. for all $x \in (0,1)$ and $t > 0$ except those satisfying $(x+ct-a)/2 \in \mathbb{Z}$, $(x+ct+a)/2 \in \mathbb{Z}$, $(x-ct-a)/2 \in \mathbb{Z}$, or $(x-ct+a)/2 \in \mathbb{Z}$. It is clear that y satisfies the wave equation for every x except for the points in the above mentioned set. To verify that y also satisfies the boundary conditions, we verify

$$y(0,t) = \frac{1}{2}(F(-ct) + F(ct)) = 0,$$
$$y(1,t) = \frac{1}{2}(F(1-ct) + F(1+ct)) = 0,$$

where the first equality holds since F is odd, and the second holds since F is 2-periodic and odd. Furthermore,

$$y(x,0) = \frac{1}{2}(F(x) + F(x)) = f(x), \qquad x \in [0,1],$$
$$y_t(x,0) = \frac{c}{2}(-F'(x) + F'(x)) = 0, \qquad x \in [0,1],$$

as required.

e) For $t = 0$ the graph of $y(x,t)$ is just the graph of f (where $a = h = 0.1$). We also know that $y(0,t) = y(1,t) = 0$ for every $t > 0$. For $t = 0.2$, we have $y(x,0.2) = (F(x+0.2) + F(x-0.2))/2$. It is clear that the graph of $F(x-0.2)$ has a corner for x in the interval $[0,1]$ when $x - 0.2 = 0.1$ (using the periodicity of F), that is when $x = 0.3$, and when $x - 0.2 = -0.1$, that is when $x = 0.1$. The graph of $F(x+0.2)$ does not have any jumps in the derivative in the interval $[0,1]$. The value of $y(x,0.2)$ at these two corner points are

$$y(0.1,0.2) = (F(0.3) + F(-0.1))/2 = (0.7/9 - 0.1)/2 = -0.1/9 \approx -0.01,$$
$$y(0.3,0.2) = (F(0.5) + F(0.1))/2 = (0.5/9 + 0.1)/2 = 0.7/9 \approx 0.78.$$

The graph is constructed by joining these known points with straight lines.
Similarly, when $t = 0.4$, we get corner points for $F(x-0.4)$ when $x - 0.4 = 0.1$ or $x - 0.4 = -0.1$, that is when $x = 0.5$ or $x = 0.3$. The values of $y(x,0.4)$ for these points are

$$y(0.3,0.4) = (F(0.7) + F(-0.1))/2 = (0.3/9 - 0.1)/2 = -1/30 \approx -0.03,$$
$$y(0.5,0.2) = (F(0.9) + F(0.1))/2 = (0.1/9 + 0.1)/2 = 1/18 \approx$$

6. a) We use separation of variables, and look for solutions of the form $u(r,t) = R(r)T(t)$. For such functions, the equation becomes

$$R(r)T'(t) = \frac{k}{r}(rR(r))''T(t).$$

If u is a nontrivial solution, we can divide by $kR(r)T(t)$ and obtain

$$\frac{T'(t)}{kT(t)} = \frac{(rR(r))''}{rR(r)}.$$

The left hand side and the right hand side are independent of r and t, respectively, and since they are equal they depend neither on r nor t, and so they are constant, say $-\lambda$. By rearranging and including the homogeneous boundary conditions, we end up with the following system of ODEs:

$$(rR(r))'' + \lambda rR(r) = 0,$$
$$T'(t) + \lambda kT(t) = 0.$$

The change of variables $X(r) = rR(r)$ transforms the first of these two equations to $X''(x) + \lambda X(x) = 0$. The appropriate boundary condition is $X(1) = bX(b) = 0$, i.e. $X(1) = X(b) = 0$. Assume first that $\lambda = \alpha^2$ for some $\alpha > 0$. If X satisfies the ODE and the first boundary condition, then

$$X(r) = B\sin(\alpha(r-1)).$$

If also the second boundary condition is satisfied, then $\alpha(b-1) = n\pi$, i.e. $\alpha = n\pi/(b-1)$. For convenience, we let $B = 1$ and let

$$R_n(r) = \frac{1}{r}\sin\left(n\pi\frac{r-1}{b-1}\right).$$

It can be checked that there are no nontrivial solutions if $\lambda \le 0$. If $\lambda = n^2\pi^2/(b-1)^2$, the equation for T has the solutions

$$T_n(t) = C\exp\left(-\frac{n^2\pi^2 k}{(b-1)^2}t\right).$$

As usual, we choose $C = 1$. A formal solution is a generalized linear combination

$$u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{r-1}{b-1}\right)\exp\left(-\frac{n^2\pi^2 k}{(b-1)^2}t\right),$$

where B_n are constants. These constants are determined by the initial condition $u(r,0) = f(r)$. We have

$$u(r,0) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{r-1}{b-1}\right) = f(r) = \frac{1}{r}rf(r).$$

The function $rf(r)$ has a Fourier sine expansion

$$rf(r) = \frac{2}{b-1}\sum_{n=1}^{\infty}\int_1^b sf(s)\sin\left(n\pi\frac{s-1}{b-1}\right)ds\sin\left(n\pi\frac{r-1}{b-1}\right)$$

on the interval $(1,b)$.

b) For this $f(r)$ we have

$$rf(r) = \sin\left(\pi\frac{r-1}{b-1}\right),$$

which is already expanded in a Fourier series (with only one term) on the interval $(1,b)$. By identifying coefficients, we see that $B_1 = 1$ and $B_n = 0$ for $n \geq 2$. From subproblem a) we then see that

$$u(r,t) = \frac{1}{r}\sin\left(\pi\frac{r-1}{b-1}\right)\exp\left(-\frac{\pi^2 k}{(b-1)^2}t\right), \qquad (1 < r < b,\ t > 0).$$