MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	Linjär analys
Avd. Matematik	7,5 hp
Examinator: Sara Maad Sasane	2010-08-13

No calculator is allowed. The problems are not arranged in order of difficulty. Result from a subproblem may be used in subsequent subproblem even if the earlier one is not attempted. Justify your answers. Grade criteria A: 90–100 points, B: 75–89 points, C: 65–74 points, D: 55–64 points, E: 50–54 points, Fx: 40–49 points, F: 0–39 points. For passing a minimum grade of E is necessary in the written exam and moreover a pass in the compulsory assessed coursework.
Good luck!

1. Let $f : (-\pi, \pi) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & -\pi < x < 0, \\ 1 & 0 \le x < \pi. \end{cases}$$

   a) Determine the Fourier series of the function f on the interval $(-\pi, \pi)$. 10 p

   b) The series of subproblem a) converges on the whole of $\mathbb{R}$ to a function F where $F(x) = f(x)$ when $x \in (-\pi, 0)$ or $x \in (0, \pi)$. Sketch the graph of F on the interval $[-5\pi, 5\pi]$. In the figure you should clearly indicate the value of F for the points $n\pi$, $n = -5, \dots, 5$. 5 p

2. The function f which is defined by

$$f(x) = \begin{cases} -1 & \text{when } -\pi < x < 0, \\ 1 & \text{when } 0 < x < \pi \end{cases}$$

   has a Fourier series

$$f(x) \sim \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin((2k+1)x).$$

   Use this fact and a theorem about integration of Fourier series to determine the Fourier series of the function

$$F(x) = \begin{cases} -x & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi. \end{cases}$$

   State why the theorem is applicable. 10 p

3. a) Find orthonormal set consisting of two polynomials of at most degree 1 on the interval $[0, 2]$ with respect to the inner product

$$(f, g) = \int_0^2 f(x)g(x)\,dx.$$

   Hint: Gram–Schmidt. 10 p

   b) Find the polynomial p of at most degree 1 which minimises the integral

$$\int_0^2 (e^x - p(x))^2\,dx.$$

   Hint: Think of an orthogonal projection. 10 p

4. Determine the Fourier sine integral of the function

$$f(x) = e^{-x}, \qquad x > 0,$$

i e, determine $B(\alpha)$ such that

$$f(x) = \int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha.$$

15 p

5. Let D_N be the Dirichlet kernel which is defined by

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx).$$

a) Show that

$$\int_0^\pi D_N(x)\,dx = \frac{\pi}{2}.$$

5 p

b) Show that

$$D_N(x) = \frac{\sin((N+1/2)x)}{2\sin(x/2)}.$$

10 p

6. a) Let u be a harmonic function in the square $0 < x < 1$, $0 < y < 1$, i e $\Delta u = 0$ in this square. On the boundary, u satisfies the conditions

$$\begin{cases} u(x,0) &= 0, \\ u(x,1) &= \sin(2\pi x), \\ u(0,y) &= 0, \\ u(1,y) &= 0. \end{cases}$$

Determine u in the whole square. 15 p

b) Let v satisfy the same equation in the square, but with the boundary conditions

$$\begin{cases} v(x,0) &= \sin(2\pi x), \\ v(x,1) &= \sin(2\pi x), \\ v(0,y) &= \sin(2\pi y), \\ v(1,y) &= \sin(2\pi y). \end{cases}$$

Determine v in the whole square. 10 p

The marked exams can be collected from 19 August from Reine Elfsö in room 208, house 6.