MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Lösningar till tentamensskrivning i
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7.5 hp
2010-08-13

1. a) The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}\pi = 1,$$

and if $n \neq 0$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx = 0, \\ b_n &= \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx = -\frac{1}{\pi n}\left((-1)^n - 1\right) = \\ &= \begin{cases} \frac{2}{\pi n} & \text{when } n \text{ is odd,} \\ 0 & \text{when } n \text{ is even.} \end{cases} \end{aligned}$$

Hence the Fourier series of f is given by

$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x).$$

b) The function F is a 2π-periodic extension of f except that in the points $n\pi$ the function has the value $1/2$. (A figure showing this is required for full marks).

2. Since f is piecewise continuous, we may integrate the series termwise. Note that

$$F(x) = \int_0^x f(t)\,dt,$$

and we integrate the given series termwise and see that

$$\begin{aligned} F(x) &\sim \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\int_0^x \sin((2k+1)t)\,dt \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\left[-\cos((2k+1)t)\right]_0^x \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\left(1 - \cos((2k+1)x)\right) \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos((2k+1)x), \end{aligned}$$

and we identify the first series in the expression with the constant $a_0/2$. Since

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) = \pi,$$

it follows that

$$F(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos((2k+1)x).$$

(The series actually converges, and so we can write =, but it was not part of the problem to verify this.)

3. a) To find an orthonormal set of polynomials of at most degree 1, we start with any set consisting of two linearly independent polynomials of degree at most 1. The polynomials $p_1(x) = 1$ and $p_2(x) = x$ will do. We construct the orthonormal basis by the Gram–Schmidt orthonormalisation method.

$$\|p_1\|^2 = (p_1, p_1) = \int_0^2 1\,dx = 2,$$

and so we let $e_1 = 1/\sqrt{2}$ be the first basis vector. It is clear that $\|e_1\| = 1$.
We calculate

$$(p_2, e_1) = \int_0^2 x\frac{1}{\sqrt{2}}\,dx = \sqrt{2},$$

and so the projection of p_2 onto e_1 is

$$(p_2, e_1)e_1 = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1.$$

A first degree polynomial which is orthogonal to e_1 is then

$$p_2 - (p_2, e_1)e_1 = x - 1,$$

and we only need to normalise this polynomial to get our second polynomial e_2:

$$\|x-1\|^2 = \int_0^2 (x-1)^2\,dx = \int_0^2 (x^2 - 2x + 1)\,dx = \cdots = \frac{2}{3},$$

and so we have $e_2 = \sqrt{\frac{3}{2}}(x-1)$.

b) Note that the integral which we need to minimize is $\|e^x - p(x)\|^2$, and so the question is which p which minimizes the norm of $e^x - p(x)$ where p is in the subspace of $L^2[0,2]$ consisting of (at most) 1st degree polynomials. Using that the distance of e^x and p is smallest when p is the orthogonal projection of e^x onto the subspace spanned by p_1 and p_2, we get

$$\begin{aligned}
p_{\min}(x) &= (e^x, p_1)p_1(x) + (e^x, p_2)p_2(x) \\
&= \int_0^2 e^x\frac{1}{\sqrt{2}}\,dx\frac{1}{\sqrt{2}} + \int_0^2 e^x\sqrt{\frac{3}{2}}(x-1)\,dx\sqrt{\frac{3}{2}}(x-1) \\
&= \frac{1}{2}(e^2-1) + 3(x-1) \\
&= 3x - \frac{7}{2} + \frac{1}{2}e^2.
\end{aligned}$$

The minimum value of the integral is then

$$\begin{aligned}
\|e^x - p_{\min}\|^2 &= \int_0^2 \left(e^{2x} + p_{\min}(x)^2 - 2e^x p_{\min}(x)\right)\,dx \\
&= \int_0^2 \left(e^{2x} + \left(9x^2 - 3(7-e^2)x + \frac{1}{4}(49 - 14e^2 + e^4)\right) - \left(6x - 7 + e^2\right)e^x\right)\,dx \\
&= \ldots
\end{aligned}$$

(The last calculation is not required for getting full marks).

4. By integration by parts, we see that

$$
\begin{aligned}
B(\alpha) &= \frac{2}{\pi}\int_0^\infty f(x)\sin(\alpha x)\,dx \\
&= \frac{2}{\pi}e^{-x}\sin(\alpha x)\,dx \\
&= \frac{2}{\pi}\left([-e^{-x}\sin(\alpha x)]_0^\infty + \int_0^\infty \alpha e^{-x}\cos(\alpha x)\,dx\right) \\
&= \frac{2}{\pi}\left(\alpha\,[-e^{-x}\cos(\alpha x)]_0^\infty - \alpha^2\int_0^\infty e^{-x}\sin(\alpha x)\,dx\right) \\
&= \frac{2}{\pi}\alpha - \alpha^2 B(\alpha).
\end{aligned}
$$

It follows that

$$B(\alpha) = \frac{2}{\pi}\frac{\alpha}{1+\alpha^2},$$

and so

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2}\sin(\alpha x)\,d\alpha.$$

5. a) Since $\int_0^\pi \cos(nx)\,dx = 0$ for $n \neq 0$, we have

$$\int_0^\pi D_N(x)\,dx = \int_0^\pi \frac{1}{2}\,dx = \frac{\pi}{2}.$$

b) Note that

$$
\begin{aligned}
D_N(x) &= \frac{1}{2} + \sum_{n=1}^{N}\cos(nx) \\
&= \frac{1}{2} + \sum_{n=1}^{N}\frac{e^{inx}+e^{-inx}}{2} \\
&= \frac{1}{2}\sum_{n=-N}^{N} e^{inx}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
2D_N(x)\sin(x/2) &= \sum_{n=-N}^{N} e^{inx}\frac{e^{ix/2}-e^{-ix/2}}{2i} \\
&= \frac{1}{2i}\sum_{n=-N}^{N}\left(e^{i(n+1/2)x} - e^{i(n-1/2)x}\right) = [\text{...telescoping series...}] \\
&= \frac{1}{2i}\left(e^{i(N+1/2)x} - e^{i(-N-1/2)x}\right) \\
&= \sin((N+1/2)x),
\end{aligned}
$$

as required.

6. a) We use separation of variables and start by looking for solutions of the form

$$u(x,y) = X(x)Y(y).$$

For such functions u, the Laplace equation $\Delta u = 0$ is equivalent to

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Dividing by $X(x)Y(y)$ (assuming that X and Y are not 0) gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

where the last equality holds since two functions which depend on two different variables (here x and y) cannot be equal unless they are constant. This gives the equations

$$\begin{aligned} X''(x) + \lambda X(x) = 0 \qquad & X(0) = X(1) = 0, \\ Y''(y) - \lambda Y(y) = 0 \qquad & Y(0) = 0, \end{aligned}$$

which have to be satisfied with the same constant λ. Solving the first of these equations, we see that after a normalization, $X(x) = \sin(n\pi x)$. For the second equation, we get $Y(y) = \sinh(n\pi y)$ (or a multiple of it).
We now make the ansatz that the solution of the problem is a series formed by such solutions:

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi y) \sin(n\pi x).$$

The last boundary condition $u(x, 1) = \sin(2\pi x)$ tells us that $b_n = 0$ for $n = 1$ and $n \geq 3$, and that $b_2 \sinh(2\pi) = 1$. This gives the solution

$$u(x, y) = \frac{1}{\sinh(2\pi)} \sin(n\pi x) \sinh(n\pi y)$$

in the square.

b) The problem can be solved without calculating so much if we note that the solution is the sum of the solutions of the four problems of the Laplace equation in the square with homogeneous boundary conditions on one side and an inhomogeneous boundary condition on the fourth side. One of these solutions in the sum is the solution that we just found in subproblem a), and so we put $u_1 = u$. Another solution in the sum is found by replacing y by $1 - y$ in the expression for u_1.

$$u_2(x, y) = \frac{1}{\sinh(2\pi)} \sin(n\pi x) \sinh(n\pi(1 - y)).$$

The other two solutions are found by changing the roles of x and y in the two solutions u_1 and u_2:

$$\begin{aligned} u_3(x, y) &= \frac{1}{\sinh(2\pi)} \sinh(n\pi x) \sin(n\pi y), \\ u_4(x, y) &= \frac{1}{\sinh(2\pi)} \sinh(n\pi(1 - x)) \sin(n\pi y). \end{aligned}$$

Adding these solutions, we get

$$v(x, y) = \frac{1}{\sinh(2\pi)} \left(\sin(n\pi x) \sinh(n\pi y) + \sin(n\pi x) \sinh(n\pi(1 - y)) + \sinh(n\pi x) \sin(n\pi y) + \sinh(n\pi(1 - x)) \sin(n\pi y) \right),$$

and it is easy to check that this function satisfies both the equation and the boundary conditions.