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| MATEMATISKA INSTITUTIONEN | Tentamensskrivning i |
| STOCKHOLMS UNIVERSITET | Linjär analys |
| Avd. Matematik | 7,5 hp |
| Examinator: Sara Maad Sasane | 2010-12-18 |

*No calculator is allowed. The problems are not arranged in order of difficulty. Results from a subproblem may be used in subsequent subproblem even if the earlier one is not attempted. Justify your answers. Grade criteria A: 90–100 points, B: 75–89 points, C: 65–74 points, D: 55–64 points, E: 50–54 points, Fx: 45–49 points, F: 0–44 points. For passing a minimum grade of E is necessary in the written exam and moreover a pass in the compulsory assessed coursework. If a student recieves the grade Fx in the written exam and has completed the assessed coursework in time, then the student will get the possibility to complete another coursework in order to get the grade E instead of Fx.*
*Good luck!*

1. Determine the Fourier series of the function $f(x) = x$ on the interval $(-1, 1)$. 8 p

2. a) Determine the cosine series for the function $f(x) = \sin(x)$ on the interval $(0, \pi)$. 7 p

   b) Show that the Fourier series of subproblem a) converges pointwise on the whole of $\mathbb{R}$ to a function $F$. 7 p

   c) Show that the Fourier series of $f$ can be differentiated pointwise, and that one then obtains the Fourier series of a function which converges to the whole of $\mathbb{R}$ to a function $G$. 7 p

   d) Sketch the graphs of $F$ and $G$ of the subproblems b) and c) on the interval $[-2\pi, 2\pi]$. Indicate in particular the values of the functions at the points of discontinuity, if any. 6 p

3. A pipe with radius 1 and length 10 is lying along the $x$-axis. We assume that the pipe has insulated faces so that the equilibrium temperature in the pipe satisfies Laplace's equation, which in cylindrical coordinates $x$ and $\theta$ is given by

$$u_{xx} + u_{\theta\theta} = 0.$$

   a) Suppose that the ends of the cylinder is kept at temperature 0 (at $x = -5$) and $f(\theta)$ (at $x = 5$), respectively. Use the method of separation of variables to determine the equilibrium temperature (i.e. the temperature after a long time) in the pipe. 15 p

   b) If the temperature is kept at $f(\theta)$ when $x = 5$ and $g(\theta)$ when $x = -5$, what will the equilibrium temperature in the pipe be? 5 p

4. The boundary value problem

$$x^2u'' + xu' + \lambda u = 0 \qquad x \in (1, c), \quad u(1) = u(c) = 0$$

   is a Sturm–Liouville problem, which can be seen by multiplying the equation with $1/x$ and rewriting the two firs terms as the derivative of a product.

   a) Determine the eigenvalues and the eigenvectors explicitly. *Hint: The equation is a Cauchy–Euler ekvation, and can therefore be solved by making the substitution $x = e^s$.* 8 p

   b) Verify that the eigenfunctions are orthogonal with respect to the inner product

$$(f, g) = \int_1^c f(x)g(x)\frac{1}{x}\,dx.$$

   Then normalize the egenfunctions to find an orthonormal set $\{\Phi_n\}_{n=1}^{\infty}$. 7 p

5. Show that

$$e^{-x^2/2} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\alpha^2/2} \cos(\alpha x)\, d\alpha, \qquad x \in \mathbb{R},$$

i.e. show that $A(\alpha) = e^{-\alpha^2/2}\sqrt{2/\pi}$ in the cosine series for $f$. *Hint: Let $f(x)$ be the right hand side of the equation above. Compute $f'(x)$ and use integration by parts to show that $f$ satisfies the differential equation $f'(x) = -xf(x)$ together with the initial condition $f(0) = 1$. Use an integrating factor to show that this initial value problem has a unique solution, $f(x) = e^{-x^2/2}$.* 15 p

6. An infinitely long wire is stretched along the $x$-axis. We assume that the temperature in the wire is described by the heat equation

$$u_t = ku_{xx}$$

with $k = 1$, and that the solution is bounded, i.e. $|u(x,t)| \le M$ for some $M > 0$ and every $x \in \mathbb{R}$, $t \ge 0$. Use separation of variables to determine the temperature for $t > 0$ in the wire if the initial temperature is given by the function $f(x) = e^{-x^2/2}$ for $x \in \mathbb{R}$. *Hint: You may use the result from problem 5.* 15 p

*The marked exams can be collected from me in room 213, house 6 on the 22 December at 13.00–13.30, and after this from Reine Elfsö in room 208, house 6.*