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STOCKHOLMS UNIVERSITET
Avd. Matematik
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Lösningar till tentamensskrivning i
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7.5 hp
2010-12-18

1. We have

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x))$$

Since f is an odd function, all the a_n's are 0, and

$$\begin{aligned} b_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx = \\ &= 2\left[-x\frac{\cos(n\pi x)}{n\pi}\right]_0^1 + 2\int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx \\ &= -2\frac{(-1)^n}{n\pi} + \frac{2}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1 \\ &= \frac{2(-1)^{n+1}}{n\pi}. \end{aligned}$$

Hence the Fourier series of f is given by

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

2. a) The cosine series of $f(x) = \sin x$ on $(0, \pi)$ is by definition

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx = -\frac{2}{\pi}(\cos(\pi) - 1) = \frac{4}{\pi}.$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos(nx)\,dx = \\ &= \frac{1}{\pi}\int_0^{\pi} \left(\sin((n+1)x) - \sin((n-1)x)\right)\,dx \\ &= -\frac{1}{\pi}\left[\frac{\cos((n+1)x)}{n+1} - \frac{\cos((n-1)x)}{n-1}\right]_0^{\pi} \\ &= -\frac{2}{\pi}\left(\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right) \\ &= -\frac{4((-1)^n+1)}{\pi(n^2-1)} = \begin{cases} -\frac{8}{\pi(n^2-1)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and so the cosine series of f on $(0, \pi)$ is given by

$$\sin(x) \sim \frac{2}{\pi} + \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{1}{4k^2-1}\cos(2kx).$$

b) f is piecewice smooth, which can be verified as follows: f is clearly piecewise continuous on $(0, \pi)$ (it is actually continuous and $f(0) = f(\pi)$, making the even 2π-periodic extension F of f a continuous function), and its derivative $f'(x) = \cos(x)$ is also piecewise continuous on this interval. By a corollary from the book, it follows that the cosine series of f converges pointwise for every $x \in \mathbb{R}$ to $(F(x+) + F(x-))/2 = F(x)$.

c) We have seen that the even periodic extension F of f is continuous and that $f'(x) = \cos(x)$ is piecewise continuous on $(0, \pi)$. Since also f' is piecewise smooth on $(0, \pi)$ (since $f''(x) = -\sin(x)$ is piecewise continuous on $(0, \pi)$, we can use a theorem in the book and conclude that the differentiated Fourier series converges on $\mathbb{R}$ to $G(x) = F'(x)$ (defined except for $x = n\pi$, $n \in \mathbb{Z}$, and so we let $G(n\pi) = 0$ (so that $G(x) = (G(x+) + G(x-))/2$ for all x) and $G(x)$ is the odd periodic extension of $\cos(x)$ on $(0, \pi)$.

d) For full marks, the functions $F(x)$ and $G(x)$ should be sketched. F is continuous, but $G(n\pi) = 0$ for $n \in \mathbb{R}$, which are the points of discontinuity.

3. a) We look for solutions of the form $u(x, \theta) = X(x)\Theta(\theta)$. From the equation we get that

$$X''(x)\Theta(\theta) + X(x)\Theta''(\theta) = 0,$$

so that if $X(x), \Theta(\theta) \neq 0$, then

$$\frac{X''(x)}{\Theta''(\theta)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda.$$

This leads to the equations

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0,$$
$$X''(x) - \lambda X(x) = 0,$$

where the first equation is equipped with the periodic boundary conditions $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$, and the second equation is equipped with the single homogeneous boundary condition $X(-5) = 0$. From boundary value problem for Θ, we se that $\lambda \geq 0$ (for $\lambda < 0$, the solutions are not periodic), and $\Theta_0(\theta) = A_0$ (for $\lambda = 0$) and $\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$ (for $\lambda = n^2$). There are no nontrivial solutions for other values of λ. For the boundary value problem for X, we obtain the solutions $X_0(x) = 1$ and $X_n(x) = \sinh(n(x + 5))$ (after ignoring constants).
A formal solution of the full boundary value problem is therefore

$$u(x, \theta) = A_0 + \sum_{n=1}^{\infty} \sinh(n(x + 5))(A_n \cos(n\theta) + B_n \sin(n\theta)),$$

where the constants A_n should be chosen such that

$$u(5, \theta) = A_0 + \sum_{n=1}^{\infty} \sinh(10n)(A_n \cos(n\theta) + B_n \sin(n\theta)) = f(\theta).$$

It follows that $A_0 = a_0/2$, $A_n = a_n/\sinh(10n)$ and $B_n = b_n/\sinh(10n)$, where a_n and b_n are the Fourier coefficients of f. Hence we obtain

$$u(x, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{\sinh(n(x + 5))}{\sinh(10n)}(a_n \cos(n\theta) + b_n \sin(n\theta)).$$

b) We first solve the same problem but with boundary conditions $u(-5, \theta) = g(\theta)$, $u(5, \theta) = 0$. The solution can be obtained from the solution of subproblem a) by replacing f by g and replacing the variable x by $-x$. We denote the solution of this problem by v. Hence

$$v(x, \theta) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \frac{\sinh(n(5 - x))}{\sinh(10n)}(\tilde{a}_n \cos(n\theta) + \tilde{b}_n \sin(n\theta)),$$

where $\tilde{a}_n$ and $\tilde{b}_n$ are the Fourier coefficients of g.
Now we denote the solution of the Laplace equation in the cylinder with the boundary conditions $u(5,\theta) = f(\theta)$ and $u(-5,\theta) = g(\theta)$ by $w(x,\theta)$. The solution w of this problem is the sum of the functions u and v, i.e.

$$w(x,\theta) = \frac{a_0 + \tilde{a}_0}{2} + \sum_{n=1}^{\infty}\Big((a_n \sinh(n(x+5)) + \tilde{a}_n \sinh(n(5-x)))\,\frac{\cos(n\theta)}{\sinh(10n)} + \left(b_n \sinh(n(x+5)) + \tilde{b}_n \sinh(n(5-x))\right)\frac{\sin(n\theta)}{\sinh(10n)}\Big)$$

4. a) Let $x = e^s$, i.e. $s = \log x$. By the chain rule we have

$$\frac{d}{dx} = \frac{ds}{dx}\frac{d}{ds} = e^{-s}\frac{d}{ds},$$
$$\frac{d^2}{dx^2} = e^{-s}\frac{d}{ds}e^{-s}\frac{d}{ds} = e^{-2s}\left(\frac{d^2}{ds^2} - \frac{d}{ds}\right).$$

When regarding u as a function of s instead of x, the Sturm–Liouville problem then transforms into the equation

$$\frac{d^2u}{ds^2} + \lambda u = 0$$

together with the boundary conditions $u(0) = u(\log c) = 0$.
We solve this problem in the usual manner, and see that $\lambda = n^2\pi^2/(\log c)^2$, $n = 1, 2, \ldots$, and

$$u_n(s) = \sin\left(\frac{n\pi s}{\log c}\right)$$

(after ignoring constants). When regarding u as a function of x instead of s, we then obtain

$$u_n(x) = \sin\left(\frac{n\pi \log x}{\log c}\right).$$

b) We need to compute

$$(u_m, u_n) = \int_1^c \sin\left(\frac{m\pi \log x}{\log c}\right)\sin\left(\frac{n\pi \log x}{\log c}\right)\frac{1}{x}\,dx.$$

We use the change of variable $x = e^s$, to change the integral into

$$(u_m, u_n) = \int_0^{\log c} \sin\left(\frac{m\pi s}{\log c}\right)\sin\left(\frac{n\pi s}{\log c}\right)ds = \frac{1}{2}\int_0^{\log c}\left(\cos\left(\frac{(m-n)\pi s}{\log c}\right) - \cos\left(\frac{(m+n)\pi s}{\log c}\right)\right)ds$$
$$= \frac{\log c}{2}\delta mn,$$

so in particular u_m and u_n are orthogonal when $m \neq n$, and the norm of u_n is $\sqrt{(\log c)/2}$. The normalized eigenfunctions are therefore

$$\Phi_n(x) = \sqrt{\frac{2}{\log c}}\sin\left(\frac{n\pi \log x}{\log c}\right).$$

5. We follow the hint, and let

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-\alpha^2/2}\cos(\alpha x)\,d\alpha.$$

Then

$$f'(x) = -\sqrt{\frac{2}{\pi}}\int_0^{\infty} \alpha e^{-\alpha^2/2}\sin(\alpha x)\,d\alpha$$
$$= \sqrt{\frac{2}{\pi}}\left[e^{-\alpha^2/2}\sin(\alpha x)\right]_0^{\infty} - \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-\alpha^2/2}x\cos(\alpha x) = -xf(x),$$

i.e. f solves the ordinary differential equation

$$f'(x) + xf(x) = 0,$$

which can be solved by multiplying with the integrating factor $e^{x^2/2}$. Indeed, the equation then becomes

$$\frac{d}{dx}\left(e^{x^2/2}f(x)\right) = 0,$$

which is equivalent to

$$f(x) = f(0)e^{-x^2/2}.$$

Moreover,

$$f(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\alpha^2/2}\,d\alpha = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}} =: \sqrt{\frac{2}{\pi}}I,$$

and

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)/2}\,dx\,dy = \frac{\pi}{2}\int_0^\infty e^{-r^2/2}r\,dr = \frac{\pi}{2},$$

and so $f(0) = 1$. This proves that

$$f(x) = e^{-x^2/2}$$

as required.

6. We use separation of variables, and look for solutions of the form $(x,t) = X(x)T(t)$. Inserting this into the equation, we get

$$X(x)T'(t) = X''(x)T(t),$$

so if $X(x)$ and $T(t)$ are not 0, then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

i.e.

$$X''(x) + \lambda X(x) = 0,$$
$$T'(t) + \lambda T(t) = 0.$$

If $\lambda = 0$, then we get the solution $X_0(x) = 1$ and $T_0(t) = 1$ (modulo multiplication with a constant). If $\lambda > 0$, then we get the solutions $X_\alpha(x) = A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)$, and $T_\alpha(t) = e^{-\alpha^2 t}$, where $\alpha > 0$. If $\lambda < 0$, then there are no bounded solutions of either of the equations.

A formal solution is given by

$$u(x,t) = A_0 + \int_0^\infty e^{-\alpha^2 t}(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\,d\alpha.$$

From the initial condition, we see that

$$e^{-x^2/2} = A_0 + \int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\,d\alpha =$$

From Problem 5, we see that we can choose $A_0 = 0$, $B(\alpha) = 0$, and

$$A(\alpha) = \sqrt{\frac{2}{\pi}}e^{-\alpha^2/2}.Hence$$

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\alpha^2 t}e^{-\alpha^2/2}\cos(\alpha x)\,d\alpha.$$

The answer can be simplified further, but this is not required for full marks.