MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sara Maad Sasane

Tentamensskrivning i
Linjär analys
7,5 hp
2011-01-10

No calculator is allowed. The problems are not arranged in order of difficulty. Results from a subproblem may be used in a subsequent subproblem even if the earlier one is not attempted. Justify your answers. Grade criteria A: 90–100 points, B: 75–89 points, C: 65–74 points, D: 55–64 points, E: 50–54 points, Fx: 45–49 points, F: 0–44 points. For passing a minimum grade of E is necessary in the written exam and moreover a pass in the compulsory assessed coursework. If a student recieves the grade Fx in the written exam and has completed the assessed coursework in time, then the student will get the possibility to complete another coursework in order to get the grade E instead of Fx.
Good luck!

1. Determine the Fourier series till of the function $f(x) = \sin(x/2)$ on the interval $(-\pi, \pi)$. 10 p

2. a) Show that the sine series of the function $f(x) = 1$ on the interval $(0, \pi)$ is

$$1 \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}, \qquad (0 < x < \pi)$$

10 p

   b) Determine the cosine series of the function $g(x) = x$ on the interval $(0, \pi)$ by integrating the series from subproblem a) termwise from 0 till x. Motivate why it is allowed to proceed like this. 10 p

   c) When you determined the constant term in b), you also determined the value of a certain number series. State this formula explicitly. 5 p

3. An inhomogeneous heat equation can be used to model heat conduction when heat is generated in a material. Here we assume that the generated heat is constant 1 throughout the material, and we study the equation in the domain $0 < x < \pi$ och $t > 0$. The equation then becomes

$$u_t = ku_{xx} + 1, \qquad (0 < x < \pi, t > 0)$$

and we assume that the end points are kept at constant temperature 0 so that we get the boundary conditions $u(0, t) = u(\pi, t) = 0$. We also assume that the temperature at $t = 0$ is $u(x, 0) = -x^2/k$, where k is the constant that occurs in the equation above.

   a) Since the equation is inhomogeneous it is not possible to directly use separation of variables in the equation above. Introduce the function $v(x, t) = u(x, t) - \Phi(x)$, and determine Φ so that the function v satisfies the homogeneous heat equation

$$v_t = kv_{xx}, \qquad (0 < x < \pi, \quad t > 0)$$

with boundary and initial conditions

$$v(0, t) = v(\pi, t) = 0,$$
$$v(x, 0) = f(x)$$

for some function f. Determine this function f. 10 p

   b) Solve the equation for v and use this solution to determine u. 10 p

4. A trigonometric polynomial is a function $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(x) = \sum_{n=1}^{k} a_n e^{i\lambda_n x}, \quad x \in \mathbb{R},$$

where $k \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{C}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The room TP of trigonometric polynomial is a vector space over $\mathbb{C}$ with respect to pointwise addition and scalar multiplication (You do not need to prove this).

a) Show that the formula

$$\langle f, g \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}dx$$

defines an inner product on TP, i.e. för f, g, $h \in TP$ and $a, b \in \mathbb{C}$ we have

(i) $\langle f, g \rangle = \overline{\langle g, f \rangle}$,

(ii) $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$,

(iii) $\langle f, f \rangle \geq 0$ with equality if and only if $f = 0$,

where $\overline{c} = a - ib$ denotes the complex conjugate of the number $c = a + ib \in \mathbb{C}$. 10 p

b) For $\lambda \in \mathbb{R}$, let $e_\lambda \in TP$ be the trigonometric polynomial

$$e_\lambda(x) = e^{i\lambda x}, \quad x \in \mathbb{R}.$$

Show that the set $\{e_\lambda : \lambda \in \mathbb{R}\}$ is orthonormal in TP. 10 p

5. Show that

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\alpha}{1+\alpha^2} \sin(\alpha x)\, dx, \qquad x > 0.$$

For full marks, a motivation why the representation is valid is required, e.g. by referring to a suitable theorem. 15 p

6. Determine a bounded solution of $\Delta u = 0$, $x, y > 0$ with the boundary values

$$u(x, 0) = e^{-x},$$
$$u(0, y) = 0.$$

Hint: It is possible to use the result of problem 5. 10 *p*

The marked exams can be collected from me in room 213, house 6 on the 14 January at 13.00–13.30, and after this from Reine Elfsö in room 208, house 6.