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Avd. Matematik
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Lösningar till tentamensskrivning i
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1. We have

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Since f is an odd function, all the a_n's are 0, and

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \sin(x/2)\sin(n\pi x)\,dx = \\
&= \frac{1}{\pi}\int_0^{\pi} \left(\cos((n-1/2)x) - \cos((n+1/2)x)\right)\,dx \\
&= \frac{1}{\pi}\left[\frac{1}{n-1/2}\sin((n-1/2)x) - \frac{1}{n+1/2}\sin((n+1/2)x)\right]_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{1}{n-1/2}\sin((n-1/2)\pi) - \frac{1}{n+1/2}\sin((n+1/2)\pi)\right) \\
&= \frac{1}{\pi}\left(\frac{1}{n-1/2}(-1)^{n+1} - \frac{1}{n+1/2}(-1)^{n}\right) \\
&= -\frac{(-1)^n}{\pi}\left(\frac{1}{n-1/2} + \frac{1}{n+1/2}\right) \\
&= -\frac{(-1)^n}{\pi}\frac{2n}{n^2-1/4} = -\frac{8n(-1)^n}{\pi(4n^2-1)}.
\end{aligned}$$

Hence the Fourier series of $\sin(x/2)$ is given by

$$\sin(x/2) \sim \frac{8n}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\sin(nx), \qquad 0 < x < \pi.$$

2. a)

$$1 \sim \sum_{n=1}^{\infty} b_n \sin(nx), \qquad 0 < x < \pi$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}\sin(nx)\,dx \\
&= \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} \\
&= -\frac{2}{n\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} \text{if } n = 2k, \\ \frac{4}{\pi(2k-1)} \quad \text{if } n = 2k-1. \end{cases}
\end{aligned}$$

b) Since $f(x) = 1$ is piecewise continuous on $(0, \pi)$ it is possible to integrate termwise (according to a theorem in the book). We then obtain

$$x = \int_0^x ds = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\int_0^x \sin((2k-1)x)\,dx$$
$$= -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}(\cos((2k-1)x)-1)$$
$$= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)x).$$

We need to determine the constant term separately. It is

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}.$$

Hence

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)x).$$

c)

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi}{2},$$

i.e.

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

3. a) We let $u(x,t) = v(x,t) + \Phi(x)$. From the equation for u we get

$$v_t = kv_{xx} + k\Phi''(x) + 1.$$

Since v should satisfy the homogeneous heat equation $v_t = kv_{xx}$, we must have $\Phi''(x) = -1/k$, i.e.

$$\Phi(x) = -\frac{1}{2k}x^2 + Ax + B,$$

for some constants A and B. The boundary conditions for v are $v(0,t) = u(0,t) - \Phi(0) = 0 - \Phi(0)$ and $v(\pi,t) = u(\pi,t) - \Phi(\pi) = 0 - \Phi(\pi)$. By demanding that the boundary conditions for v are homogeneous, we get $\Phi(0) = \Phi(\pi) = 0$. These conditions determine the constants A and B, i.e. $A = \pi/(2k)$ and $B = 0$. We conclude that the function Φ is given by

$$\Phi(x) = -\frac{1}{2k}x^2 + \frac{\pi}{2k}x.$$

The initial condition for v is

$$v(x,0) = u(x,0) - \Phi(x) = -\frac{1}{k}x^2 + \frac{1}{2k}x^2 - \frac{\pi}{2k}x = -\frac{1}{2k}x^2 - \frac{\pi}{2k}x = f(x).$$

b) We use separation of variables, and look for solutions of the form $v(x,t) = X(x)T(t)$. Then

$$X(x)T'(t) = kX''(x)T(t),$$

and if $X(x)$, $T(t)$ are not 0, then

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

i.e. we get the two equations

$$X''(x) + \lambda X(x) = 0,$$
$$T'(t) + \lambda k T(t) = 0$$

with boundary conditions $X(0) = X(\pi) = 0$ for the first equation. This equation has solutions for $\lambda = n^2$, $n = 1, 2, 3, \ldots$ which are

$$X_n(x) = \sin(nx)$$

and the second equation with $\lambda = n^2$ has solutions

$$T_n(t) = e^{-n^2 kt}.$$

By superposition we get a formal solution

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-n^2 kt}.$$

The constants are determined so that the initial condition $v(x, 0) = f(x)$ is satisfied, i.e.

$$v(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = -\frac{1}{2k}x^2 - \frac{\pi}{2k}x,$$

i.e. B_n should be the n:th Fourier sine coefficients of f on the interval $(0, \pi)$. The Fourier sine coefficients of f have to be computed, and we start by computing the coefficients for x and x^2. By linearity the sine coefficients of f are the sum of the coefficients of these two functions multiplied by the constants $-\pi/(2k)$ and $-1/(2k)$, respectively. The n:th sine coefficient of x is

$$\frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\left[-x\frac{\cos(nx)}{n}\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos(nx)\,dx = -2\frac{(-1)^n}{n},$$

and the n:th sine coefficient of x^2 is

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx &= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos(nx)\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{2x}{n}\cos(nx)\,dx \\ &= -2\pi\frac{(-1)^n}{n} + \frac{4}{n^2\pi}\left[x\sin(nx)\right]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\sin(nx)\,dx \\ &= -2\pi\frac{(-1)^n}{n} + \frac{4}{n^3\pi}((-1)^n - 1).\end{aligned}$$

Hence the n:th sine coefficient of f is

$$B_n = \frac{\pi}{2k}2\frac{(-1)^n}{n} - \frac{1}{2k}\left(-2\pi\frac{(-1)^n}{n} + \frac{4}{n^3\pi}((-1)^n - 1)\right) = \frac{2\pi}{k}\frac{(-1)^n}{n} - \frac{2}{kn^3\pi}((-1)^n - 1).$$

It now follows that

$$v(x,t) = \frac{1}{k}\sum_{n=1}^{\infty}\left(2\pi\frac{(-1)^n}{n} - \frac{2}{n^3\pi}((-1)^n - 1)\right)\sin(nx).$$

Hence

$$u(x,t) = v(x,t) + \Phi(x) = \frac{1}{k}\sum_{n=1}^{\infty}\left(2\pi\frac{(-1)^n}{n} - \frac{2}{n^3\pi}((-1)^n - 1)\right)\sin(nx) - \frac{1}{2k}x^2 + \frac{\pi}{2k}x$$

4. a) We verify (i)–(iii) as follows:

(i) $\langle f, g\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\overline{g(x)}\,dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\overline{g(x)\overline{f(x)}}\,dx = \overline{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)\overline{f(x)}\,dx} = \overline{\langle g, f\rangle}.$

(ii) $\langle af+bg,h\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(af(x)+bg(x))\overline{h(x)}\,dx = a\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)\overline{h(x)}\,dx+b\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(x)\overline{h(x)}\,dx = a\langle f,h\rangle + b\langle g,h\rangle$.

(iii) $\langle f,f\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{n=1}^{k}\sum_{m=1}^{k}a_n\overline{a_m}e^{i\lambda_n x}e^{-i\lambda_m x}\,dx = \lim_{T\to\infty}\frac{1}{2T}\sum_{n=1}^{k}\sum_{m=1}^{k}a_n\overline{a_m}\int_{-T}^{T}e^{i(\lambda_n-\lambda_m)x}\,dx = \lim_{T\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}a_n\overline{a_m}\delta_{nm}$ (see calculation in b below) $= \sum_{n=1}^{k}|a_n|^2 \geq 0$ and if equality holds then $a_n = 0$ for every $n \in \{1,\dots,k\}$, hence $f = 0$.

b) We need to prove that $\langle e_\lambda, e_\mu\rangle = 0$ if $\lambda \neq \mu$ and $\langle e_\lambda, e_\lambda\rangle = 1$. If $\lambda \neq \mu$, then

$$|\langle e_\lambda, e_\mu\rangle| = |\langle e^{i\lambda x}, e^{i\mu x}\rangle| = \lim_{T\to\infty}\left|\int_{-T}^{T}e^{i(\lambda-\mu)x}\,dx\right| = \lim_{T\to\infty}\frac{1}{2T|\lambda-\mu|}|e^{i(\lambda-\mu)T}-e^{-i(\lambda-\mu)T}| \leq \lim_{T\to\infty}\frac{2}{2T|\lambda-\mu|} = 0,$$

while

$$\langle e_\lambda, e_\lambda\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dx = 1.$$

5.

$$e^{-x} = \int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha,$$

where

$$\begin{aligned} B(\alpha) &= \frac{2}{\pi}\int_0^\infty e^{-x}\sin(\alpha x)\,dx \\ &= \frac{2}{\pi}\operatorname{Im}\int_0^\infty e^{-x}e^{i\alpha x}\,dx \\ &= \frac{2}{\pi}\operatorname{Im}\int_0^\infty e^{(-1+i\alpha)x}\,dx \\ &= \frac{2}{\pi}\operatorname{Im}\left(\frac{1}{-1+i\alpha}\left[e^{(-1+i\alpha)x}\right]_0^\infty\right) \\ &= \frac{2}{\pi}\operatorname{Im}\left(\frac{1}{1-i\alpha}\right) \\ &= \frac{2}{\pi}\operatorname{Im}\frac{1+i\alpha}{1+\alpha^2} = \frac{2}{\pi}\frac{\alpha}{1+\alpha^2}. \end{aligned}$$

Hence

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2}\sin(\alpha x)\,d\alpha, \qquad (x>0),$$

and the representation is valid since $\int_0^\infty |e^x|\,dx = \int_0^\infty e^x\,dx < \infty$ and e^{-x} is piecewise smooth on every bounded interval of $\mathbb{R}_+$, and since $f(x) = e^{-x}$ is continuous on $\mathbb{R}_+$, we clearly have $f(x) = (f(x+)+f(x-))/2$ for every point $x > 0$.

6. We use separation of variables, and look for solutions $u(x,y) = X(x)Y(y)$, x, $y > 0$. From the equation we get

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

If $X(x)$ and $Y(y) \neq 0$ this gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda,$$

for some constant λ, i.e.

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ Y''(y) - \lambda Y(y) &= 0, \end{aligned}$$

and X has the boundary condition $X(0) = 0$. Here $X(x) = B\sin(\alpha x)$, where $\alpha > 0$, and we take $B = 1$. $Y(y) = Ae^{\alpha y} + Be^{-\alpha y}$, and $A = 0$ since we are looking for bounded solutions. By superposition we get a formal solution

$$u(x,y) = \int_0^\infty B(\alpha)\sin(\alpha x)e^{-\alpha y}\,d\alpha,$$

and since

$$u(x,0) = \int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha,$$

$B(\alpha)$ is the $B(\alpha)$ in the Fourier sine integral representation of e^{-x}, i.e.

$$B(\alpha) = \frac{2}{\pi}\frac{\alpha}{1+\alpha^2}.$$

Hence

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2}\sin(\alpha x)e^{-\alpha y}\,d\alpha.$$