MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Annemarie Luger

Exam in
Linear analysis,
january 2, 2012

No calculator, book or notes are allowed. Give **complete justifications** *for your answers!*
At least 15 points are needed in order to pass the exam!

1. Consider the function $f(x) = 1$ on $[0, \pi]$.

   (a) Determine the sine series $\sum_{n=1}^{\infty} b_n \sin nx$ of f. (3p)

   (b) Set $g(x) := \sum_{n=1}^{\infty} b_n \sin nx$ for $x \in \mathbb{R}$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at the jumping points! (1p)

2. Consider the function $F(x) = \begin{cases} 0 & -\pi \leq x \leq -\frac{\pi}{2} \\ \cos x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq x \leq \pi \end{cases}$.

   (a) Determine the Fourier series of F and show

   $$F(x) = \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k2-1}\cos 2kx. \qquad (4p)$$

   Hint: $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$.

   (b) Use the above result in order to calculate the following sums:

   $$\sum_{k=1}^{\infty} \frac{(-1)^k}{4k2-1} \qquad \sum_{k=1}^{\infty} \frac{1}{4k2-1} \qquad \sum_{k=1}^{\infty} \frac{1}{(4k2-1)2} \qquad \sum_{m=1}^{\infty} \frac{(-1)^m}{16m2-1}. \qquad (6p)$$

3. Consider the Sturm-Liouville problem

   $$y'' + \lambda y = 0, \qquad y(0) + y'(0) = 0,\ y(\pi) + y'(\pi) = 0. \qquad (1)$$

   (a) Determine the eigenvalues and eigenfunctions of (1). (4p)

   (b) The function $\varphi(x) = x2 - (\pi+2)x + \pi + 2$ satisfies the boundary conditions in (1). Write down the expansion of φ in eigenfunctions of (1). Give the formulas for the coefficients (but you do not need to calculate them!!!). (1p)

4. Solve the boundary value problem (5p)

   $$\begin{aligned} &u_t - u_{xx} = 0 \\ &u(0,t) = u_x(1,t) = 0 \qquad\qquad x \in [0,1],\ t \in [0,\infty). \\ &u(x,0) = \sin \tfrac{3\pi}{2} x \end{aligned}$$

Please turn!

5. (a) Are

$$c_n = \frac{n2+i}{n2+1},\ n \in \mathbb{Z}$$

the complex Fourier coefficients of a function $h \in E$? Explain! (2p)

(b) Consider the linear space $V = C[-2,2]$ of continous functions $f : [-2,2] \to \mathbb{C}$. Does

$$\langle f, g\rangle := \int_{-2}^{2} f(x)\overline{g(x)}(1 - x2)\,dx$$

define an inner product on V? If yes, prove it! If no, give a counter example! (2p)

(c) Show: If $f, f' \in C(\mathbb{R})$ and $f, f', f'' \in G$ then there exists C such that

$$|\mathcal{F}[f](\omega)| \le \frac{C}{|\omega|^2}.$$

Hint: You may use that the Fourier transform $\mathcal{F}[g]$ is bounded for $g \in G$. (2p)

The marked exams can be collected from me in room 209, house 6, on 10/1 at 10:30–11:00, and after this from Lisa Källström, room 204. If you want to get your result by email, please send me a message (with your name only) to `luger@math.su.se`

Good luck!