MATEMATISKA INSTITUTIONEN
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Avd. Matematik
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Solutions to the exam in
Linear analysis,
january 2, 2012

Note that these are complete solutons but only commented answers!!!

1. Consider the function $f(x) = 1$ on $[0, \pi]$.

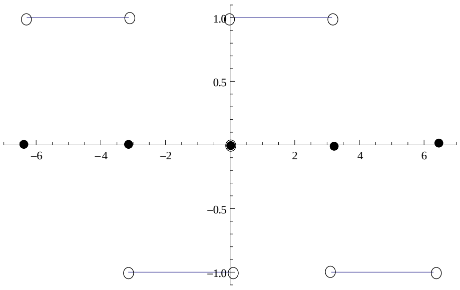
   (a) We obtain

$$b_n = \frac{2}{\pi}\int_0^\pi 1 \cdot \sin nx\, dx = \ldots = \frac{2}{\pi n}\big((-1)^{n+1} + 1\big) = \begin{cases} 0 & n = 2k \\ \frac{4}{\pi(2k+1)} & n = 2k+1 \end{cases}$$

   and hence

$$f \sim \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi}{2k+1}.$$

   (b) Note that g is even, 2π-periodic, and $g(k\pi) = 0$ for $k \in \mathbb{Z}$. Graph of g:



2. (b) Note that F satisfies the conditions of Dirichlets theorem:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2 - 1} = \underline{\frac{1}{2} - \frac{\pi}{4}} \qquad \text{use } x = 0$$

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \underline{\frac{1}{2}} \qquad \text{use } x = \frac{\pi}{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(4k^2 - 1)2} = \underline{\frac{\pi^2}{16} - \frac{1}{2}} \qquad \text{use Parsevals identity}$$

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{16m^2 - 1} = \underline{\frac{1}{2} - \frac{\sqrt{2}\pi}{8}} \qquad \text{use } x = \frac{\pi}{4}$$

3. (a) We get

$$\lambda_n = n^2 \text{ with eigenfunction } y_n(x) = -n\cos nx + \sin nx \qquad \text{for } n = 1, 2, \ldots$$

   and

$$\lambda_0 = -1 \text{ with eigenfunction } y_0(x) = e^{-x}.$$

   (b) It holds

$$\varphi(x) = c_0 e^{-x} + \sum_{n=1}^{\infty} c_n(-n\cos nx + \sin nx)$$

   where

$$c_n = \frac{\int_0^\pi \varphi(x) y_n(x)\, dx}{\int_0^\pi |y_n(x)|^2\, dx} \text{ for } n = 0, 1, 2, \ldots.$$

4. Separation of variables leads to the Sturm-Liouville problem

$$X'' - \lambda X = 0 \quad \text{with} \quad X(0) = X'(1) = 0.$$

According to the Lemma we see immediately that there are no negative eigenvalues. Otherwise we obtain

$$X_n(x) = \sin(2n+1)\frac{\pi}{2}x \quad \text{for } n = 0, 1, 2 \ldots$$

which leads to the ansatz

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin(2n+1)\frac{\pi}{2}x.$$

Using the differential equation and the initial conditions gives the following equations for the coefficients u_n

$$\left.\begin{aligned} \dot{u_n} + \left((2n+1)\tfrac{\pi}{2}\right)^2 u_n = 0 \\ u_n(0) = 0 \end{aligned}\right\} \quad \text{for } n \neq 1 \quad \text{and} \quad \left.\begin{aligned} \dot{u_1} + \left(\tfrac{3\pi}{2}\right)^2 u_1 = 0 \\ u_1(0) = 1 \end{aligned}\right\} \quad \text{for } n = 1.$$

This gives

$$u_n(t) \equiv 0 \quad \text{for } n \neq 0 \qquad \text{and } u_1(t) = e^{-\frac{9\pi^2}{4}t}$$

and hence finally

$$u(x,t) = e^{-\frac{9\pi^2}{4}t} \sin\frac{3\pi}{2}x.$$

5. (a) The sequence

$$c_n = \frac{n^2 + i}{n^2 + 1}, \; n \in \mathbb{Z}$$

is **not** a sequence of complex Fourier coefficients of a function $h \in E$, since $c_n \not\to 0$ (should hold according to Riemann-Lebesque-Lemma).

(b) $\langle f, g \rangle$ does **not** define an inner product as eg. for $f(x) \equiv 1$ it holds $\langle f, f \rangle = -\frac{4}{3} < 0$.

(c) According to the assumption the hint can be applied to f'', and hence there exists a constant C such that

$$|\mathcal{F}[f'']| \leq C.$$

As

$$\mathcal{F}[f''] = (i\omega)^2 \mathcal{F}[f]$$

the claim follows.