MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Exam in
Linear analysis,
January 28, 2012

No calculator, book or notes are allowed. Give **complete justifications** *for your answers!*
At least 15 points are needed in order to pass the exam!

1. Consider the function
$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| < \pi \end{cases}.$$

   (a) Determine the Fourier series of f, namely $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.
   Hint: Please find the list of trigonometric formulae on the backside of this sheet! (3p)

   (b) Set $g(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ for $x \in \mathbb{R}$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at the jumping points! (1p)

   (c) Give an example of an interval [a,b] where the Fourier series of f converges uniformly. Give an example of an interval [c,d] where the Fourier series of f does not converge uniformly.
   Explain! (1p)

2. Consider the function $F(x) = \dfrac{e^x + e^{-x}}{e^{\pi} - e^{-\pi}}$ on $[-\pi, \pi]$.

   (a) Determine the complex Fourier series of F. (3p)

   (b) Show
$$F(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos nx. \qquad \text{(1p)}$$

   (c) Use the above result in order to calculate the following sums:
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{1+n^2} \qquad \sum_{m=1}^{\infty} (-1)^m \frac{(2m+1)}{1+(2m+1)^2}. \qquad \text{(5p)}$$

3. Consider the Sturm-Liouville problem
$$y'' + \lambda y = 0, \qquad y(0) = 0,\ y(\pi) - y'(\pi) = 0. \qquad (1)$$

   (a) Determine the eigenvalues and eigenfunctions of (1).
   Remark: In case that the eigenvalues can not be calculated explicitely, draw a picture indicating where they are situated approximately. (4p)

   (b) Formulate the orthogonality property of the eigenfunctions and check it directly. (1p)

4. Solve the boundary value problem (5p)
$$\begin{aligned} &u_t - 2u_{xx} = 0 \\ &u_x(0,t) = u_x(\pi,t) = 0 \qquad x \in [0,\pi],\ t \in [0,\infty). \\ &u(x,0) = \cos^2 x \end{aligned}$$

Please turn!

5. (a) Is $\{e^{7inx}; n \in \mathbb{Z}\}$ a complete orthogonal system in E? Explain! (2p)

(b) Let $h \in E$ be given. What can be said about h if

$$\int_{-\pi}^{\pi} h(x)e^{inx}\,dx = \int_{-\pi}^{\pi} h(x)e^{-inx}\,dx = 0 \quad \text{for } n = 1, 2, 3, \ldots$$

holds? (2p)

(c) Show: If $f \in G(\mathbb{R})$ is real-valued then it holds $\mathcal{F}[f](-\omega) = \overline{\mathcal{F}[f](\omega)}$. (2p)

The marked exams can be collected from me in room 209, house 6, on 3/2 at 10:30–11:00, and after this from Lisa Källström, room 204. If you want to get your result by email, please send me a message (with your name only) to `luger@math.su.se`

Good luck!

3. Useful Trigonometric Formulae

$$\sin(\pi - \alpha) = \sin\alpha \qquad \sin n\pi = 0, \quad n \in \mathbb{Z}$$

$$\cos(\pi - \alpha) = -\cos\alpha \qquad \cos n\pi = (-1)^n, \quad n \in \mathbb{Z}$$

$$\sin\alpha = \cos(\tfrac{\pi}{2} - \alpha) \qquad \tan\alpha = \cot(\tfrac{\pi}{2} - \alpha)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin^2\alpha = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\alpha \qquad \cos^2\alpha = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\alpha$$

$$\sin^3\alpha = \tfrac{1}{4}(3\sin\alpha - \sin 3\alpha) \qquad \cos^3\alpha = \tfrac{1}{4}(3\cos\alpha + \cos 3\alpha)$$

$$\sin^4\alpha = \tfrac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha) \qquad \cos^4\alpha = \tfrac{1}{8}(3 + 4\cos 2\alpha + \cos 4\alpha)$$