MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Exam in
Linear analysis,
August 17, 2012

No calculator, book or notes are allowed. Give **complete justifications** *for your answers!*
At least 15 points are needed in order to pass the exam!

1. Consider the function
$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| < \pi \end{cases}.$$

   (a) Determine the Fourier series of f, namely $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.
   Hint: Please find the list of trigonometric formulae on the backside of this sheet! (3p)

   (b) Set $g(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ for $x \in \mathbb{R}$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at the jumping points! (1p)

   (c) Give an example of an interval [a,b] where the Fourier series of f converges uniformly. Give an example of an interval [c,d] where the Fourier series of f does not converge uniformly.
   Explain! (1p)

2. Consider the function $F(x) = x^2$ on $[-\pi, \pi]$.

   (a) Determine the cosine series of F. (3p)

   (b) Show
$$F(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad (1p)$$

   (c) Use the above result in order to calculate the following sums:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}. \qquad (5p)$$

3. Consider the Sturm-Liouville problem
$$(x^3 y')' + \lambda x y = 0, \qquad y(1) = y(e) = 0. \qquad (1)$$

   (a) Determine the eigenvalues and eigenfunctions of (1).
   Remark: Use the substitution $x = e^t$. (4p)

   (b) Which orthogonality properties do the eigenfunctions satisfy? (1p)

4. Solve the boundary value problem (5p)
$$\begin{aligned} &u_{tt} = u_{xx}, \quad 0 \le x \le \pi,\ t \ge 0 \\ &u(0,t) = u_x(\pi,t) = u(0,t) \\ &u_t(x,0) = \sin\tfrac{x}{2} + 2\sin\tfrac{5x}{2} \end{aligned}.$$

Please turn!

5. (a) Find two (real valued) polynomials of at most degree 1 on the interval $[-1,1]$ that are orthonormal with respect to the inner product

$$(f,g)=\int_{-1}^{1} f(x)g(x)\,dx. \tag{2}$$

Is this problem uniquely solvable? How many solutions might one expect? Only one has to be given! (2p)

(b) Find a polynomial p of at most degree 1 which minimizes the integral

$$\int_{-1}^{1} (e^x - p(x))^2\,dx. \qquad (2p)$$

(c) Is (2) an inner product on the space of piecewise continuous functions? Justify your answer! (2p)

The marked exams can be collected from me in room 209, house 6, on 21/8 at 10:30–11:00, and after this from Lisa Källström, room 204. If you want to get your result by email, please send me a message (with your name only) to `luger@math.su.se`

Good luck!

3. Useful Trigonometric Formulae

$$\sin(\pi-\alpha)=\sin\alpha \qquad \sin n\pi = 0, \quad n\in\mathbb{Z}$$

$$\cos(\pi-\alpha)=-\cos\alpha \qquad \cos n\pi=(-1)^n, \quad n\in\mathbb{Z}$$

$$\sin\alpha=\cos(\tfrac{\pi}{2}-\alpha) \qquad \tan\alpha=\cot(\tfrac{\pi}{2}-\alpha)$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha} \qquad \cot\alpha=\frac{\cos\alpha}{\sin\alpha}$$

$$\sin^2\alpha+\cos^2\alpha=1$$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$$

$$\sin\alpha\sin\beta=\tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta=\tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta=\tfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\sin^2\alpha=\tfrac{1}{2}-\tfrac{1}{2}\cos 2\alpha \qquad \cos^2\alpha=\tfrac{1}{2}+\tfrac{1}{2}\cos 2\alpha$$

$$\sin^3\alpha=\tfrac{1}{4}(3\sin\alpha-\sin 3\alpha) \qquad \cos^3\alpha=\tfrac{1}{4}(3\cos\alpha+\cos 3\alpha)$$

$$\sin^4\alpha=\tfrac{1}{8}(3-4\cos 2\alpha+\cos 4\alpha) \qquad \cos^4\alpha=\tfrac{1}{8}(3+4\cos 2\alpha+\cos 4\alpha)$$