MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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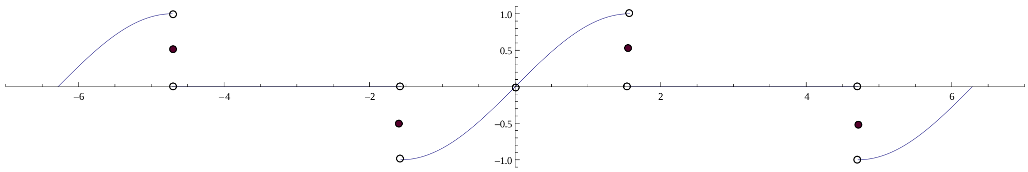
Solutions to the exam in
Linear analysis,
August 17, 2012

Note that these are not complete solutions but only commented answers!!!

1. (a) $f \sim \frac{1}{2}\sin x + \sum_{l=1}^{\infty} \frac{4l}{\pi}\frac{(-1)^{l+1}}{4l^2-1}\sin 2lx$

   (b) Note that g is even, 2π-periodic, and $g(\frac{2k+1}{\pi}) = \frac{1}{2}(-1)^k$ for $k \in \mathbb{Z}$.
   Graph of g:



   (c) The Fourier series of f converges uniformly eg. on $[-\frac{\pi}{4}, \pi 4]$ as f is smooth on this interval,.
   it does not converge uniformly eg on $[0, \pi]$, as f has a jump.

2. Note that F satisfies the conditions of Dirichlets theorem.

   (c)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \underline{\frac{\pi^2}{12}} \qquad \text{use } x = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \underline{\frac{\pi^2}{6}} \qquad \text{use } x = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \underline{\frac{\pi^4}{90}} \qquad \text{use Parsevals identity}$$

3. Consider the Sturm-Liouville problem

$$(x^3y')' + \lambda xy = 0, \qquad y(1) = y(e) = 0. \tag{1}$$

   (a) Determine the eigenvalues and eigenfunctions of (1).
   Remark: Use the substitution $x = e^t$. (4p)

   (b) Which orthogonality properties do the eigenfunctions satisfy? (1p)

4. Solve the boundary value problem (5p)

$$u_{tt} = u_{xx}, \quad 0 \le x \le \pi,\ t \ge 0$$
$$u(0,t) = u_x(\pi,t) = u(0,t) \qquad .$$
$$u_t(x,0) = \sin\tfrac{x}{2} + 2\sin\tfrac{5x}{2}$$

Please turn!

5. (a) Find two (real valued) polynomials of at most degree 1 on the interval $[-1,1]$ that are orthonormal with respect to the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,dx.$$

Is this problem uniquely solvable? How many solutions might one expect? Only one has to be given! (2p)

(b) Find a polynomial p of at most degree 1 which minimizes the integral

$$\int_{-1}^{1} (e^x - p(x))^2\,dx.$$

Hint: Think of an orthogonal projection. (2p)

The marked exams can be collected from me in room 209, house 6, on 21/8 at 10:30–11:00, and after this from Lisa Källström, room 204. If you want to get your result by email, please send me a message (with your name only) to `luger@math.su.se`

Good luck!

3. Useful Trigonometric Formulae

$\sin(\pi - \alpha) = \sin\alpha$ $\qquad$ $\sin n\pi = 0, \quad n \in \mathbb{Z}$

$\cos(\pi - \alpha) = -\cos\alpha$ $\qquad$ $\cos n\pi = (-1)^n, \quad n \in \mathbb{Z}$

$\sin\alpha = \cos(\frac{\pi}{2} - \alpha)$ $\qquad$ $\tan\alpha = \cot(\frac{\pi}{2} - \alpha)$

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ $\qquad$ $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos 2\alpha$ $\qquad$ $\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$

$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$ $\qquad$ $\cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$

$\sin^4\alpha = \frac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha)$ $\qquad$ $\cos^4\alpha = \frac{1}{8}(3 + 4\cos 2\alpha + \cos 4\alpha)$