MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Exam in
Linear analysis,
January 17, 2013

No calculator, book or notes are allowed. Give **complete justifications** *for your answers!*

At this written exam you can earn at most 75 points, at least 50 points are needed in order to pass the exam! If you have passed the written part you may do a (non-obligatory) oral exam in order to earn additional (at most) 25 points. The grading is based on both parts, that is 100 possible points. If you want to do the oral exam, write an e-mail to me (luger@math.su.se) within one week!

1. Consider the function (15p)

$$f(x) = \begin{cases} 1 - x^2 & -\pi \leq x < 0 \\ 1 + x^2 & 0 \leq x < \pi \end{cases}.$$

   (a) Determine the Fourier series of f, namely $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

   (b) Set $g(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ for $x \in \mathbb{R}$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at the jump points!

2. Consider the function $F(x) = \cos \frac{x}{2}$ on $[-\pi, \pi]$. (15p)

   (a) Determine the Fourier series of F.
   Hint: You may use the list of trigonometric formulae on the backside of this sheet!

   (b) Show

$$F(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} \cos nx$$

   (c) Use the above result in order to calculate values of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \qquad \sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2} \qquad \sum_{m=1}^{\infty} \frac{(-1)^m}{16m^2 - 1}.$$

3. Let the function h be defined by (2p)

$$h(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}, \qquad x \in \mathbb{R}.$$

   Is h continuous? Explain your answer!

4. Let the Fourierseries of a function $f \in E'$ be given by (3p)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

   What is the value of $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ in terms of f?

5. Solve the boundary value problem (25p)

$$u_t - u_{xx} = 3, \quad 0 \leq x \leq 1, \, t \geq 0$$
$$u_x(0,t) = 0, \quad u_x(1,t) = -2$$
$$u(x,0) = \cos 5\pi x - x^2.$$

   Hint: Which boundary values has the function $h(x,t) = -x^2$?

Please turn!

6. Consider the following boundary value problem on the halv plane (15p)

$$(\star) \quad \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad x \in \mathbb{R}, \quad y > 0. \\ u(x,0) = \dfrac{1}{1+x^2}, \; x \in \mathbb{R}. \end{cases}$$

(a) Calculate the Fourier transform of $f(x) = e^{-|x|}$ and use your result in order to obtain the Fourier transform of $g(x) = \frac{1}{1+x^2}$.

(b) Denote the Fourier transform of $u(x, y)$ with respect to the variable x by $U(\omega, y)$. If $u(x, y)$ is a solution to the boundary value problem $(\star)$ which boundary value problem does $U(\omega, y)$ satisfy? .

The marked exams can be collected from me in room 209, house 6, on 28/1 at 16:00–16:30, and after this from Katarina Ringels, room 204. If you want to get your result by email, please send me a message (with your name only) to `luger@math.su.se`

Good luck!

3. Useful Trigonometric Formulae

$\sin(\pi - \alpha) = \sin\alpha \qquad \sin n\pi = 0, \quad n \in \mathbb{Z}$

$\cos(\pi - \alpha) = -\cos\alpha \qquad \cos n\pi = (-1)^n, \quad n \in \mathbb{Z}$

$\sin\alpha = \cos(\frac{\pi}{2} - \alpha) \qquad \tan\alpha = \cot(\frac{\pi}{2} - \alpha)$

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} \qquad \cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos 2\alpha \qquad \cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$

$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \qquad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$

$\sin^4\alpha = \frac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha) \qquad \cos^4\alpha = \frac{1}{8}(3 + 4\cos 2\alpha + \cos 4\alpha)$