MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Annemarie Luger

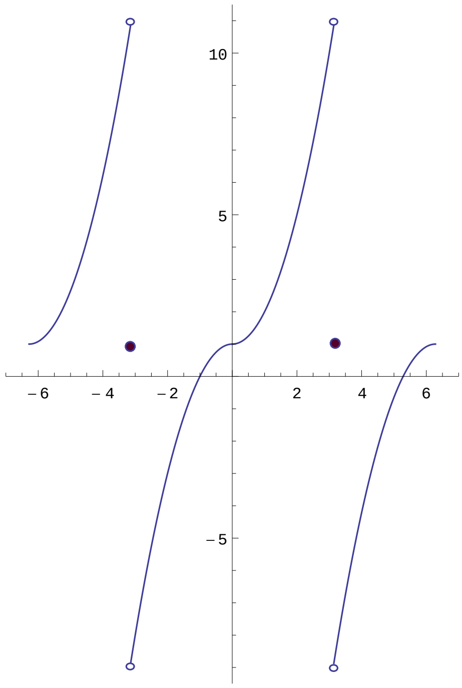
Solutions to the exam in
Linear analysis,
January 17, 2013

*Note that these are not complete solutions but only commented answers!!!*

1. (a) Note that $f(x) = 1 + f_1(x)$, where $f_1$ is odd.

$$f \sim 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2\pi^2 - 2) - 2}{n^3} \sin nx$$

   (b) Graph of $g$:




2. (a)

$$F \sim \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} \cos nx.$$

   (b) $F(x)$ equals its Fourier series, as $F \in E$, $F$ continous and $F(-\pi) = F(\pi)$.

   (c) $x = 0$ gives: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \underline{\frac{2-\pi}{4}}$

   Parsevals identity gives: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2} = \underline{\frac{\pi^2 - 8}{16}}$

   $x = \frac{\pi}{2}$ gives $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{16m^2 - 1} = \underline{\frac{2\sqrt{2} - \pi}{4\sqrt{2}}}.$

3. Yes, $h$ is continuous, since there exists a $x$-independent convergent majorant:

$$\sum_{n=1}^{\infty} \left| \frac{\sin nx}{n^3} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

and hence the series converges uniformly. This means that $h$ is the uniform limit of continous functions and hence continous.

4. $\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty} a_n \cos nx$ is the *cos*-series of $f$, or in other words the Fourserier series of the even part of $f$, that is $f_{even}(x) := \frac{f(x)+f(-x)}{2}$. As $f \in E'$ also $f_{even} \in E'$. Then Dirichlets Theorem gives:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{f_{even}(x+) + f_{even}(x-)}{2} = \frac{f(x+) + f(-x+) + f(x-) + f(-x-)}{4}$$

5.

$$\begin{aligned} &u_t - u_{xx} = 3, \quad 0 \le x \le 1,\, t \ge 0 \\ &u_x(0,t) = 0, \quad u_x(1,t) = -2 \\ &u(x,0) = \cos 5\pi x - x^2. \end{aligned}$$

As the function $h(x,t) = -x^2$ satisfies the inhomogenious boundary conditions, we set $u(x,t) = v(x,t) - x^2$, and observe that $v$ satisfies the following boundary value problem:

$$\begin{aligned} &v_t - v_{xx} = 1, \quad 0 \le x \le 1,\, t \ge 0 \\ &v_x(0,t) = 0, \quad v_x(1,t) = 0 \\ &v(x,0) = \cos 5\pi x. \end{aligned}$$

The eigenfunctions $X(x)$ of the problem satisfy:

$$\frac{X''}{X} = -\lambda \qquad X'(0) = X'(1) = 0$$

This gives: $X_n(x) = \cos n\pi x$ for $n = 0, 1, 2, \ldots$. (NOTE $n = 0$ is included here!)

The Ansatz $v(x,t) = \sum\limits_{n=0}^{\infty} T_n(t) \cos n\pi x$ leads to

$$\dot{T}_0 = 1 \quad \text{ and } \quad \dot{T}_n + n^2\pi^2 T_n = 0 \text{ for } n \ge 1$$

and

$$T_5(0) = 1 \quad \text{ and } \quad T_n(0) = 0 \text{ for } n \ne 5.$$

Solving these initial value problems leads to $v(x,t) = t + e^{-(5\pi)^2 t} \cos 5\pi x$ and hence

$$\underline{u(x,t) = t + e^{-(5\pi)^2 t} \cos 5\pi x - x^2}.$$

6. (a) Direct calculation gives

$$(\mathcal{F}[f])(\omega) = \frac{1}{\pi(1+\omega^2)}.$$

Use then $\mathcal{F}[\mathcal{F}[f]](x) = \dfrac{1}{2\pi} f(-x)$ to obtain

$$\mathcal{F}[\frac{1}{1+x^2}] = \frac{1}{2} e^{-|\omega|}.$$

(b)

$$\begin{aligned} &(i\omega)^2 U + U_{yy} = 0 \\ &U(\omega, 0) = \frac{1}{2} e^{-|\omega|} \end{aligned}$$