MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Annemarie Luger

Exam in
Linear analysis,
February 16, 2013

No calculator, book or notes are allowed. Give **complete justifications** *for your answers!*

At this written exam you can earn at most 75 points, at least 50 points are needed in order to pass the exam! If you have passed the written part you may do a (non-obligatory) oral exam in order to earn additional (at most) 25 points. The grading is based on both parts, that is 100 possible points. If you want to do the oral exam, write an e-mail to me (luger@math.su.se) within one week!

1. Consider the function (15p)

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \cos x & 0 \le x < \pi \end{cases}.$$

   (a) Determine the Fourier series of f, namely $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.
   Hint: You may use the list of trigonometric formulae on the backside of this sheet!

   (b) Set $g(x) := \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ for $x \in \mathbb{R}$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at the jump points!

2. Consider the function $F(x) = x$ on $]-\pi, \pi[$. (15p)

   (a) Determine the *complex* Fourier series of F

   (b) Show

$$F(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx$$

   (c) Use the above result in order to calculate values of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

3. Solve the boundary value problem (25p)

$$\begin{aligned} &u_{tt} - u_{xx} = x, \quad 0 \le x \le \pi,\ t \ge 0 \\ &u(0,t) = 0, \quad u_x(\pi, t) = 0 \\ &u(x,0) = \sin \tfrac{3x}{2}, \quad u_t(x,0) = 0 \end{aligned}$$

4. (a) Show the modulation formula for the Fourier transform, namely, it holds (15p)

$$\mathcal{F}[f(x)\cos cx](\omega) = \frac{1}{2}\Big(\mathcal{F}[f](\omega - c) + \mathcal{F}[f](\omega + c)\Big), \qquad \text{for all } f \in G(\mathbb{R}) \text{ and } c \in \mathbb{R}.$$

   (b) Calculate the Fourier transform of

$$f(x) = \begin{cases} \cos \pi x & |x| \le 1 \\ 0 & |x| > 1 \end{cases}.$$

5. Denote by $\big(y_n(x)\big)_{n\in\mathbb{N}}$ the orthonormal system of eigenfunctions of the following Sturm Liouville-problem (2p)

$$\begin{aligned} &-(e^x y')' + y \arctan x = \lambda \tfrac{1}{1+x^2}\, y, \quad -1 \le x \le 3 \\ &y'(-1) = 0, \quad y(3) = 0. \end{aligned}$$

   Write down the orthonormality conditions (as integrals).

Please turn!

6. Let $V = C^1[0,2]$ be the space of complex valued continously differentiable functions defined on the interval $[0,2]$. Which of the following define an inner product on V, and which do not? Explain! (3p)

(a) $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$.

(b) $\langle f, g\rangle = \int_0^2 f^2(x)\overline{g^2(x)}\,dx$.

(c) $\langle f, g\rangle = \int_0^2 f'(x)\overline{g'(x)}\,dx$.

(d) $\langle f, g\rangle = f(1)\overline{g(1)} + \int_0^2 f'(x)\overline{g'(x)}\,dx$.

The marked exams can be collected from me in room 209, house 6, on 19/2 at 10:00–10:30, and after this from Katarina Ringels, room 204. If you want to get your result by email, please send me a message (with your name only) to `luger@math.su.se`

Good luck!

3. Useful Trigonometric Formulae

$\sin(\pi - \alpha) = \sin\alpha \qquad \sin n\pi = 0, \quad n \in \mathbb{Z}$

$\cos(\pi - \alpha) = -\cos\alpha \qquad \cos n\pi = (-1)^n, \quad n \in \mathbb{Z}$

$\sin\alpha = \cos(\frac{\pi}{2} - \alpha) \qquad \tan\alpha = \cot(\frac{\pi}{2} - \alpha)$

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} \qquad \cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos 2\alpha \qquad \cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$

$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \qquad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$

$\sin^4\alpha = \frac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha) \qquad \cos^4\alpha = \frac{1}{8}(3 + 4\cos 2\alpha + \cos 4\alpha)$