MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Solutions to the exam in
February 16, 2013

Note that these are not complete solutions but only commented answers!!!

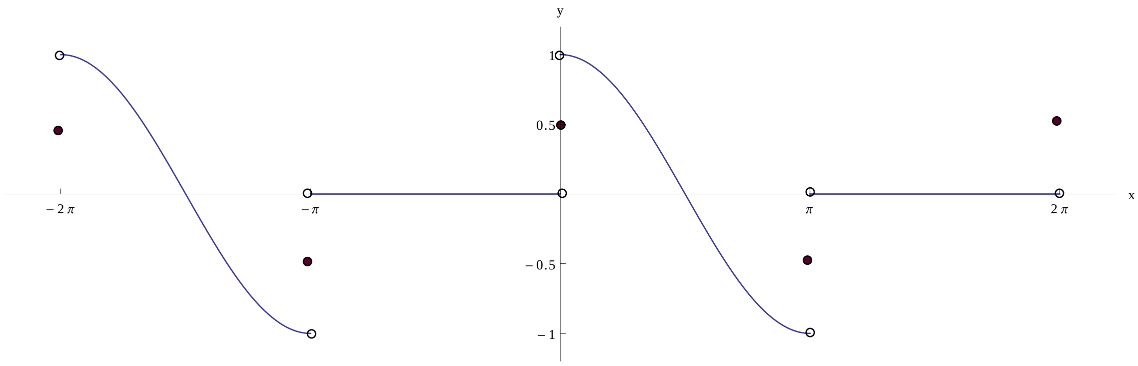
1. (a) Direct calculation gives

$$a_n = \begin{cases} \frac{1}{2} & n = 1 \\ 0 & n \neq 1 \end{cases} \qquad \text{and} \qquad b_n = \begin{cases} 0 & n = 1 \\ \frac{(-1)^n+1}{\pi} \cdot \frac{n}{n^2-1} & n \neq 1 \end{cases}.$$

and hence

$$f \sim \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{\pi} \cdot \frac{n}{n^2-1} \sin nx = \frac{1}{2}\cos x + \sum_{k=1}^{\infty} \frac{4k}{\pi(4k^2-1)} \sin 2kx$$

(b) Graph of g:



2. (a)

$$F \sim \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{i(-1)^n}{n} e^{inx}.$$

(b) Rewriting the complex Fourier series as the real Fourier series gives

$$F \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx.$$

$F(x)$ equals its Fourier series on the open intervall, as $F \in E$, F continous on $]-\pi, \pi[$.

(c) $x = \frac{\pi}{2}$ gives: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \underline{\frac{\pi}{4} - 1}$

Parsevals identity gives: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \underline{\frac{\pi}{6}}$

3.

$$\begin{aligned} &u_{tt} - u_{xx} = x, \quad 0 \leq x \leq \pi,\ t \geq 0 \\ &u(0,t) = 0, \quad u_x(\pi, t) = 0 \\ &u(x,0) = \sin \tfrac{3x}{2}, \quad u_t(x,0) = 0 \end{aligned}$$

The eigenfunctions $X(x)$ of the problem satisfy:

$$\frac{X''}{X} = -\lambda \qquad X(0) = X'(\pi) = 0$$

This gives: $X_n(x) = \sin \frac{2n+1}{2} x$ for $n = 0, 1, 2, \ldots$.

Expanding the function x (right handside of the PDE) with respect to the eigenfunctions gives:

$$x \sim \sum_{n=0}^{\infty} c_n \sin\frac{2n+1}{2}x, \qquad \text{where } c_n = \frac{\int_0^\pi x \sin\frac{2n+1}{2}x\,dx}{\int_0^\pi \sin^2\frac{2n+1}{2}x\,dx} = \frac{8(-1)^n}{\pi(2n+1)^2}.$$

The Ansatz $u(x,t) = \sum\limits_{n=0}^{\infty} T_n(t)\sin\frac{2n+1}{2}x$ leads to

$$\ddot{T}_n + \Big(\frac{2n+1}{2}\Big)^2 T_n = c_n$$

$$T_n(0) = \begin{cases} 0 & n \neq 1 \\ 1 & n = 1 \end{cases} \qquad \text{and} \qquad \dot{T}_n(0) = 0 \qquad \text{for } n = 0,1,2,\dots$$

Solving these initial value problems leads to

$$T_n(x) = A_n \cos\frac{2n+1}{2}t, \qquad \text{where} \begin{cases} A_1 = 1 + \frac{2}{\pi} & \\ A_n = \frac{2(-1)^{n+1}}{\pi} & n \neq 1 \end{cases}$$

and hence

$$u(x,t) = -\frac{2}{\pi}\cos\frac{t}{2}\sin\frac{x}{2} + \Big(1+\frac{2}{\pi}\Big)\cos\frac{3}{2}t\sin\frac{3}{2}x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{\pi}\cos\frac{2n+1}{2}t\,\sin\frac{2n+1}{2}x.$$

4. (a) The modulation formula for the Fourier transforms is shown by direct calculation.

   (b) One way to solve the problem is using (a) and the auxiliary function $h(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$.

   for which it holds $(\mathcal{F}[h])(\omega) = \frac{\sin\omega}{\pi\omega}$. Then it follows

$$(\mathcal{F}[f])(\omega) = \frac{\omega\sin\omega}{\pi(\pi^2-\omega^2)}.$$

5. The weight function in this problem is $\varrho(x) = \frac{1}{1+x^2}$ and hence the orthogonality conditions become

$$\int_{-1}^{3} \frac{y_n(x)\overline{y_m(x)}}{1+x^2}\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}.$$

6. (a) No, $\langle f,f\rangle = 0$ is possible even if f is not the zero function, namely if the support of f is contained in the interval $[1,2]$.

   (b) No, e.g. in general it does not hold $\langle \lambda f, g\rangle = \lambda\langle f,g\rangle$.

   (c) No, $\langle f,f\rangle$ is possible even if f is not the zero function, namely if f is constant.

   (d) Yes, properties 1, 3, an 4 are obvious.
   Let us check 2: $\langle f,f\rangle = |f(1)|^2 + \int_0^2 |f'(x)|^2\,dx = 0$ implies $f(1) = 0$ and as f' continous also $f'(x) \equiv 0$. Hence f is constant and by $f(1) = 0$ it follows $f \equiv 0$.