STOCKHOLM UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Written exam
Linear analysis
January 17, 2015

No calculator, book or notes are allowed. Complete justifications of your answers should be included. The problems are not arranged in order of difficulty. Results from a subproblem may be used in a subsequent subproblem even if the earlier subproblem has not been solved.

To pass the exam, the sum of the obtained C- and T-points should be no less than 15. Detailed information regarding the grading is found at the course web page. Optional oral exams will take place during the period January 26–January 30 for students that have scored at least 11 C-points and at least 11 T-points on this exam. Students that wish to take the oral exam should send an e-mail to thomas.onskog@math.su.se no later than on January 23.

1. Consider the function (3C 2T)

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}.$$

   (a) Determine the sine series of f on $[0, \pi]$.

   (b) Let g be the sine series of f on $[0, \pi]$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at jumping points.

   (c) Let h be the Fourier series of f on $[-\pi, \pi]$. Sketch the graph of h on the interval $[-2\pi, 2\pi]$. Be particularly clear at jumping points.

   (d) Give an example of a closed interval $[a, b]$, such that h converges uniformly on $[a, b]$, whereas g does not converge uniformly on $[a, b]$. Motivate your answer.

2. Consider the function (3C 2T)

$$f(x) = \begin{cases} \sin 2x, & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0, & \text{for all other } x \in [-\pi, \pi]. \end{cases}$$

   (a) Determine the Fourier series of f on $[-\pi, \pi]$.

   (b) Determine the Fourier series of f' on $[-\pi, \pi]$.

   (c) Calculate the sums

$$\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2 (2k+1)^2} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)}{(2k-3)(2k+1)}.$$

3. Determine all eigenvalues of the Sturm-Liouville problem (3C 2T)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & \text{for } 0 < x < 1, \\ X(0) - X'(0) = X(1) = 0. \end{cases}$$

   Find the corresponding orthonormal system of eigenfunctions.

4. Solve the boundary value problem (3C 2T)

$$\begin{cases} u_t - u_{xx} = 0, & \text{for } 0 < x < \pi, t > 0, \\ u(0,t) = 0,\ u(\pi,t) = 1, & \text{for } t > 0, \\ u(x,0) = x, & \text{for } 0 < x < \pi. \end{cases}$$

Please turn over!

5. (a) Calculate the Fourier transform of (3C 2T)

$$f(x) = xe^{-|x|}.$$

(b) Calculate the Fourier transform of

$$g(x) = \frac{x}{\left(x^2+1\right)^2}.$$

(c) Use Plancherel's identity to calculate the definite integral

$$\int_0^{\infty} \frac{x^2}{\left(1+x^2\right)^4}\,dx.$$

6. Let f be a twice continuously differentiable function on $\mathbb{R}$. (5T)

(a) Show that if f is odd, then so is f''.

(b) Find all odd f such that $f''(x) \geq 0$ holds for all x (that is all odd, convex functions).

(c) Show that if f is $2c$-periodic, then $\int_{-c}^{c} f''(x)\,dx = 0$.

(d) Find all periodic f such that $f''(x) \geq 0$ holds for all x (that is all periodic, convex functions).

The marked exams can be collected in room 406, house 6 on January 21 at 15:30-16:00, and after that at the Student Affairs office, rooms 203–204, house 6. To get the exam result by e-mail, please send a message to thomas.onskog@math.su.se.

Good luck!

Useful trigonometric formulae

$$\sin(\pi-\alpha) = \sin\alpha, \qquad \sin n\pi = 0, \text{ for } n \in \mathbb{Z},$$
$$\cos(\pi-\alpha) = -\cos\alpha, \qquad \cos n\pi = (-1)^n, \text{ for } n \in \mathbb{Z},$$
$$\sin\alpha = \cos\left(\frac{\pi}{2}-\alpha\right), \qquad \tan\alpha = \cot\left(\frac{\pi}{2}-\alpha\right),$$
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha},$$
$$\sin^2\alpha + \cos^2\alpha = 1,$$
$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha,$$
$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right),$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right),$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right),$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2},$$
$$\sin^3\alpha = \frac{3\sin\alpha - \sin 3\alpha}{4}, \qquad \sin^3\alpha = \frac{3\cos\alpha + \cos 3\alpha}{4},$$
$$\sin^4\alpha = \frac{3-4\cos 2\alpha + \cos 4\alpha}{8}, \qquad \sin^4\alpha = \frac{3+4\cos 2\alpha + \cos 4\alpha}{8},$$