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Written exam
Linear analysis
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1. Consider the function (3C 2T)

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}.$$

(a) Determine the sine series of f on $[0, \pi]$.
Solution: The sine series of f on $[0, \pi]$ is defined as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

To determine b_n, we first use partial integration to calculate the integral

$$\begin{aligned} I &= \int_0^{\pi} e^x \sin nx dx = \left[e^x \frac{-\cos nx}{n} \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} e^x \cos nx dx \\ &= -e^{\pi} \frac{\cos n\pi}{n} + e^0 \frac{\cos 0}{n} + \frac{1}{n} \left(\left[e^x \frac{\sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} e^x \frac{\sin nx}{n} dx \right) = \frac{1}{n} \left(1 - e^{\pi}(-1)^n\right) - \frac{1}{n^2} I, \end{aligned}$$

implying

$$I = \frac{\frac{1}{n}\left(1 - e^{\pi}(-1)^n\right)}{1 + \frac{1}{n^2}} = \frac{n\left(1 - e^{\pi}(-1)^n\right)}{1 + n^2}.$$

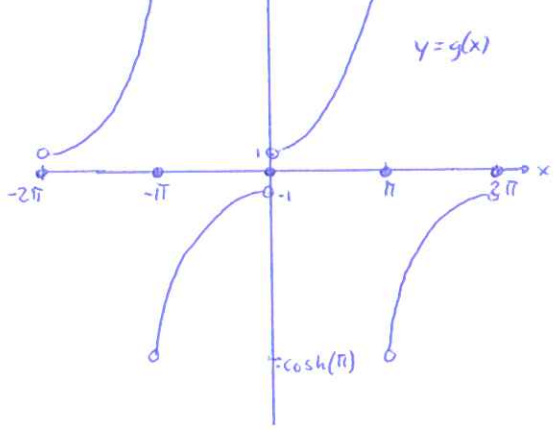
Similarly

$$\int_0^{\pi} e^{-x} \sin nx dx = \frac{n\left(1 - e^{-\pi}(-1)^n\right)}{1 + n^2},$$

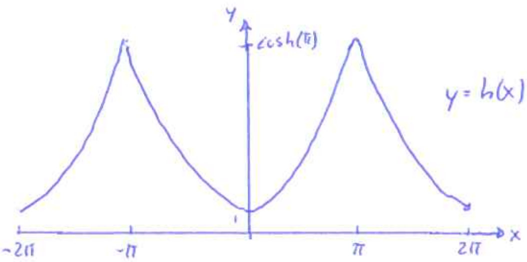
so

$$b_n = \frac{2}{\pi} \frac{1}{2} \left(\frac{n\left(1 - e^{\pi}(-1)^n\right)}{1 + n^2} + \frac{n\left(1 - e^{-\pi}(-1)^n\right)}{1 + n^2} \right) = \frac{2n\left(1 - (-1)^n \cosh \pi\right)}{\pi\left(1 + n^2\right)}$$

(b) Let g be the sine series of f on $[0, \pi]$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at jumping points.
Solution: For $x \in (-\pi, 0)$, g is given by the odd extension of f from $(0, \pi)$, so that $g(x) = -f(-x)$. Since f is a continuously differentiable function, it satisfies the prerequisites of Dirichlet's theorem and, at $x = 0$ and $x = \pm\pi$, g equals the mean value of its left and right limits which is zero. Moreover, g is 2π-periodic. Below, we sketch the graph of g on $[-2\pi, 2\pi]$.



(c) Let h be the Fourier series of f on $[-\pi, \pi]$. Sketch the graph of h on the interval $[-2\pi, 2\pi]$. Be particularly clear at jumping points.
Solution: The function h is a 2π-periodic extension of f from $(-\pi, \pi)$ to the entire real line. Since f is a continuously differentiable function, it satisfies the prerequisites of Dirichlet's theorem and, since $f(\pi) = f(-\pi)$, h will be the continuous function sketched below



(d) Give an example of a closed interval $[a, b]$, such that h converges uniformly on $[a, b]$, whereas g does not converge uniformly on $[a, b]$. Motivate your answer.
Solution: Since f and f' belongs to E, any closed interval containing at least one of the discontinuities of g will satisfy the required conditions.

2. Consider the function (3C 2T)

$$f(x) = \begin{cases} \sin 2x, & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0, & \text{for all other } x \in [-\pi, \pi]. \end{cases}$$

(a) Determine the Fourier series of f on $[-\pi, \pi]$.
Solution: f is an odd functions, so the terms a_n will be zero for all $n \geq 0$. It suffices to determine b_n and we first suppose that $n \neq 2$. In that case

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin 2x \sin nx dx \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (\cos(n-2)x - \cos(n+2)x)\, dx = \frac{1}{2\pi}\left[\frac{\sin(n-2)}{n-2} - \frac{\sin(n+2)}{n+2}\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi}\left(\frac{\sin(n-2)\pi/2}{n-2} - \frac{\sin(n+2)\pi/2}{n+2}\right) = \frac{1}{\pi}\left(\frac{\sin(2k-3)\pi/2}{2k-3} - \frac{\sin(2k+1)\pi/2}{2k+1}\right) \\ &= \frac{(-1)^k}{\pi}\left(\frac{1}{2k-3} - \frac{1}{2k+1}\right) = \frac{4(-1)^k}{\pi(2k-3)(2k+1)}, \end{aligned}$$

if we use the substitution $n = 2k - 1$. In the case $n = 2$, we get

$$b_2 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2 2x dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1+\cos 4x}{2} dx = \frac{1}{\pi}\left[\frac{x}{2} + \frac{\sin 4x}{8}\right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{2},$$

so that the requested Fourier series is

$$f(x) \sim \frac{1}{2}\sin 2x + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-3)(2k+1)} \sin(2k-1)x.$$

(b) Determine the Fourier series of f' on $[-\pi, \pi]$.
Solution: Since f is continuous on $[-\pi, \pi]$, $f(\pi) = f(-\pi)$ and $f' \in E$, the Fourier series of f is termwise differentiable. Hence

$$f'(x) \sim \cos 2x + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)}{(2k-3)(2k+1)} \cos(2k-1)x.$$

(c) Calculate the sums

$$\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2 (2k+1)^2} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)}{(2k-3)(2k+1)}.$$

Solution: Applying Parseval's identity to the Fourier series of f, we get

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right) = |b_2|^2 + \sum_{k=1}^{\infty} |b_{2k-1}|^2 \\ &= \frac{1}{4} + \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-3)^2 (2k+1)^2}. \end{aligned}$$

We already know that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin^2 2x dx = \frac{1}{2},$$

so the first sum equals

$$\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2 (2k+1)^2} = \frac{\pi^2}{16} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi^2}{64}.$$

Moreover, since f' is continuously differentiable and f' is continuous at zero, it follows from Dirichlet's theorem that

$$f'(0) = 1 + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)}{(2k-3)(2k+1)},$$

and since $f'(0) = 2$, we immediately get

$$\sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)}{(2k-3)(2k+1)} = \frac{\pi}{4}.$$

3. Determine all eigenvalues of the Sturm-Liouville problem (3C 2T)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & \text{for } 0 < x < 1, \\ X(0) - X'(0) = X(1) = 0. \end{cases}$$

Find the corresponding orthonormal system of eigenfunctions.
Solution: The eigenvalues are real, so we investigate the three cases: $\lambda = -\omega^2 < 0$, $\lambda = 0$ and $\lambda = \omega^2 > 0$. In the case of negative eigenvalues, the general solution is

$$X(x) = ae^{\omega x} + be^{-\omega x}.$$

The second boundary condition gives

$$X(1) = ae^{\omega} + be^{-\omega} = 0 \quad \Rightarrow \quad b = -ae^{2\omega},$$

and hence $X(x) = a\left(e^{\omega x} - e^{\omega(2-x)}\right)$ with derivative $X'(x) = a\omega\left(e^{\omega x} + e^{\omega(2-x)}\right)$. The first boundary condition now gives

$$X(0) - X'(0) = a\left(1 - e^{2\omega}\right) - a\omega\left(1 + e^{2\omega}\right) = 0.$$

A non-trivial solution would then satisfy

$$\omega = \frac{1 - e^{2\omega}}{1 + e^{2\omega}} = -\tanh\omega,$$

which is impossible since $\tanh\omega > 0$ for $\omega > 0$. Note that this case could also have been excluded using the lemma on non-negativity of eigenvalues of certain Sturm-Liouville problems. In the case of zero eigenvalues, the general solution is

$$X(x) = ax + b.$$

The boundary conditions give $X(0) - X'(0) = b - a = 0$ and $X(1) = a + b = 0$, which has the trivial solution only. In the case of positive eigenvalues, the general solution is

$$X(x) = a\cos\omega x + b\sin\omega x.$$

The first boundary condition gives

$$X(0) - X'(0) = a - b\omega = 0 \quad \Rightarrow \quad a = b\omega,$$

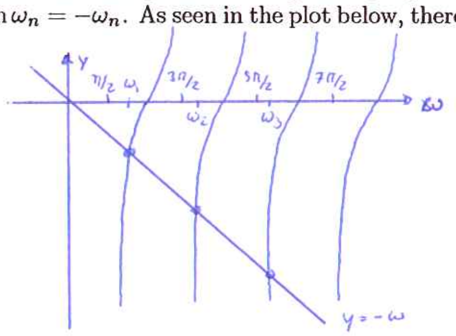
and hence $X(x) = b(\omega\cos\omega x + \sin\omega x)$. The second boundary condition now gives

$$X(1) = b(\omega\cos\omega + \sin\omega) = 0,$$

so that a non-trivial solution must satisfy

$$\omega\cos\omega = -\sin\omega \quad \Rightarrow \quad \tan\omega = -\omega.$$

The eigenvalues to the Sturm-Liouville problem are given by $\lambda_n = \omega_n^2$, where ω_n are the positive solutions to the equation $\tan\omega_n = -\omega_n$. As seen in the plot below, there are infinitely many eigenvalues.



The corresponding eigenfunctions are

$$\begin{aligned} X_n(x) &= A_n(\omega_n\cos\omega_n x + \sin\omega_n x) = A_n(\sin\omega_n x - \tan\omega_n\cos\omega_n x) \\ &= A_n\frac{\cos\omega_n\sin\omega_n x - \sin\omega_n\cos\omega_n x}{\cos\omega_n} = A_n\frac{\sin(x-1)\,\omega_n}{\cos\omega_n}. \end{aligned}$$

We calculate

$$\begin{aligned}\int_0^1 \sin^2(x-1)\,\omega_n dx &= \int_0^1 \frac{1-\cos 2(x-1)\,\omega_n}{2}dx = \left[\frac{x}{2} - \frac{\sin 2(x-1)\,\omega_n}{4\omega_n}\right]_0^1 = \frac{1}{2} - \frac{\sin 2\omega_n}{4\omega_n} \\ &= \frac{1}{2} - \frac{\sin\omega_n\cos\omega_n}{2\omega_n} = \frac{1}{2}\left(1+\cos^2\omega_n\right),\end{aligned}$$

and find that the norms of the eigenfunctions are given by

$$A_n = \sqrt{\frac{2\cos^2\omega_n}{1+\cos^2\omega_n}} = \sqrt{\frac{2}{\frac{1}{\cos^2\omega_n}+1}} = \sqrt{\frac{2}{1+\tan^2\omega_n+1}} = \sqrt{\frac{2}{2+\omega_n^2}},$$

so that the normalized eigenfunctions are

$$X_n(x) = \sqrt{\frac{2}{2+\omega_n^2}}\left(\omega_n\cos\omega_n x + \sin\omega_n x\right).$$

4. Solve the boundary value problem (3C 2T)

$$\begin{cases} u_t - u_{xx} = 0, & \text{for } 0<x<\pi, t>0, \\ u(0,t)=0,\ u(\pi,t)=1, & \text{for } t>0, \\ u(x,0)=x, & \text{for } 0<x<\pi. \end{cases}$$

Solution: First note that the function $h(x,t) = \frac{x}{\pi}$, satisfies the boundary conditions $h(0,t)=0$ and $h(\pi,t)=1$. Let $u(x,t)=v(x,t)+h(x,t)$. Then v satisfies the boundary value problem

$$\begin{cases} v_t - v_{xx} = 0, & \text{for } 0<x<\pi, t>0, \\ v(0,t)=0,\ v(\pi,t)=1, & \text{for } t>0, \\ v(x,0)=x\left(1-\frac{1}{\pi}\right), & \text{for } 0<x<\pi. \end{cases}$$

Separating the variables as $v(x,t)=X(x)T(t)$, we get the Sturm-Liouville problem

$$\begin{cases} X''(x)+\lambda X(x)=0, & \text{for } 0<x<\pi, \\ X(0)=X(\pi)=0. \end{cases}$$

and the differential equation $T'(t)+\lambda T(t)=0$. By the lemma on non-negativity of solutions to certain Sturm-Liouville problems, we know that the possibilities are either $\lambda=\omega^2>0$ or $\lambda=0$. The case $\lambda=0$ corresponds to $X(x)=ax+b$, but an application of the boundary conditions immediately gives that only the trivial solution can exist in this case. In the case $\lambda>0$, the general solution is

$$X(x) = a\cos\omega x + b\sin\omega x.$$

The first boundary condition gives $a=0$ and the second boundary condition then gives $b\sin\omega=0$, which has nontrivial solutions for integer values of ω. Hence, all eigenvalues have the form $\lambda=n^2$ and the corresponding eigenfunctions are $X_n(x)=\sin nx$. The general solution to the differential equation $T_n'(t)+\lambda_n T_n(t)=0$ is

$$T_n(t) = c_n e^{-n^2 t},$$

and we put

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx.$$

The initial condition now gives

$$x\left(\frac{\pi-1}{\pi}\right)=v(x,0)=\sum_{n=1}^{\infty}c_n\sin nx.$$

The Fourier series of the function $f(x)=x$ is given by

$$x\sim\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx,$$

so

$$c_n=\frac{2(\pi-1)(-1)^{n+1}}{\pi n},$$

and

$$u(x,t)=v(x,t)+h(x,t)=\frac{2(\pi-1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\sin nx+\frac{x}{\pi}.$$

5. (a) Calculate the Fourier transform of (3C 2T)

$$f(x)=xe^{-|x|}.$$

Solution: By definition of the Fourier transform, we get

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}xe^{-|x|}e^{-i\omega x}dx=\frac{1}{2\pi}\left(\int_{-\infty}^{0}xe^{x}e^{-i\omega x}dx+\int_{0}^{\infty}xe^{-x}e^{-i\omega x}dx\right)\\
&= \frac{1}{2\pi}\left(\left[x\frac{e^{x(1-i\omega)}}{1-i\omega}\right]_{-\infty}^{0}-\int_{-\infty}^{0}\frac{e^{x(1-i\omega)}}{1-i\omega}dx+\left[x\frac{e^{x(-1-i\omega)}}{-1-i\omega}\right]_{0}^{\infty}-\int_{0}^{\infty}\frac{e^{x(-1-i\omega)}}{-1-i\omega}dx\right)\\
&= -\frac{1}{2\pi}\left(\left[\frac{e^{x(1-i\omega)}}{(1-i\omega)^2}\right]_{-\infty}^{0}+\left[\frac{e^{x(-1-i\omega)}}{(-1-i\omega)^2}\right]_{0}^{\infty}\right)=\frac{1}{2\pi}\left(-\frac{1}{(1-i\omega)^2}+\frac{1}{(-1-i\omega)^2}\right)\\
&= \frac{(1-i\omega)^2-(1+i\omega)^2}{2\pi\left((1-i\omega)(1+i\omega)\right)^2}=\frac{-4i\omega}{2\pi\left(1-i^2\omega^2\right)^2}=-\frac{2i\omega}{\pi\left(1+\omega^2\right)^2}.
\end{aligned}$$

(b) Calculate the Fourier transform of

$$g(x)=\frac{x}{(x^2+1)^2}.$$

Solution: By the formula for inverse Fourier transformation, we get

$$xe^{-|x|}=\text{P.V.}\int_{-\infty}^{\infty}-\frac{2i\omega}{\pi\left(1+\omega^2\right)^2}e^{i\omega x}d\omega.$$

Changing the names of the variables, we get

$$\begin{aligned}
\omega e^{-|\omega|} &= \text{P.V.}\int_{-\infty}^{\infty}-\frac{2iy}{\pi\left(1+y^2\right)^2}e^{iy\omega}dy=[x=-y]=\text{P.V.}\int_{-\infty}^{\infty}\frac{2ix}{\pi\left(1+x^2\right)^2}e^{-i\omega x}dx\\
&= 4i\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x}{\left(1+x^2\right)^2}e^{-i\omega x}dx\right),
\end{aligned}$$

since the integral on the right hand side exists. This proves that

$$G(\omega) = \frac{1}{4i}\omega e^{-|\omega|} = -\frac{i}{4}\omega e^{-|\omega|}.$$

(c) Use Plancherel's identity to calculate the definite integral

$$\int_0^\infty \frac{x^2}{(1+x^2)^4}dx.$$

Solution: Plancherel's identity states that

$$\frac{1}{2\pi}\int_{-\infty}^\infty |f(x)|^2\,dx = \int_{-\infty}^\infty |F(\omega)|^2\,d\omega,$$

and by the formula in b we have

$$\begin{aligned}\int_0^\infty \frac{x^2}{(1+x^2)^4}dx &= \frac{1}{2}\int_{-\infty}^\infty \left(\frac{x}{(x^2+1)^2}\right)^2 dx = \pi\int_{-\infty}^\infty \left|-\frac{i}{4}\omega e^{-|\omega|}\right|^2 d\omega = \frac{\pi}{16}\int_{-\infty}^\infty \omega^2 e^{-2|\omega|}d\omega \\ &= \frac{\pi}{8}\int_0^\infty \omega^2 e^{-2\omega}d\omega = \frac{\pi}{8}\left(\left[\omega^2\frac{e^{-2\omega}}{-2}\right]_0^\infty - \int_0^\infty 2\omega\frac{e^{-2\omega}}{-2}d\omega\right) \\ &= \frac{\pi}{8}\int_0^\infty \omega e^{-2\omega}d\omega = \frac{\pi}{8}\left(\left[\omega\frac{e^{-2\omega}}{-2}\right]_0^\infty - \int_0^\infty \frac{e^{-2\omega}}{-2}\right) = \frac{\pi}{16}\left[\frac{e^{-2\omega}}{-2}\right]_0^\infty = \frac{\pi}{32}.\end{aligned}$$

6. Let f be a twice continuously differentiable function on $\mathbb{R}$. (5T)

(a) Show that if f is odd, then so is f''.
Solution: Since f is odd, we have that $f(x) = -f(-x)$. Hence, using the substitution $k = -h$, we get

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{k\to 0}\frac{-f(x+k)+f(x)}{-k} = f'(x),$$

so f' is an even function. Similarly, the derivative of an even function is an odd function and hence, $f'' = (f')'$ must be an odd function.

(b) Find all odd f such that $f''(x) \geq 0$ holds for all x (that is all odd, convex functions).
Solution: If f is odd, so is f'', so f'' is an odd, non-negative function. The only possible sunch function is the zero function, so f'' is identically zero. Hence $f(x) = ax+b$. Since f is odd, we also get

$$-(a(-x)+b) = ax+b \quad\Rightarrow\quad b=0,$$

so all odd, convex functions have the form $f(x) = ax$.

(c) Show that if f is $2c$-periodic, then $\int_{-c}^c f''(x)\,dx = 0$.
Solution: The derivative of a $2c$-periodic function is $2c$-periodic. Hence, by the fundamental theorem of calculus, we have

$$\int_{-c}^c f''(x)\,dx = f'(c) - f'(-c) = f'(c) - \acute{f}(-c+2c) = 0.$$

(d) Find all periodic f such that $f''(x) \geq 0$ holds for all x (that is all periodic, convex functions).
Solution: By the prerequisites f'' is a non-negative function satisfying $\int_{-c}^c f''(x)\,dx$ for some c. Since f'' is continuous, it must then be identically zero. Hence $f(x) = ax+b$. Since f is periodic, we also get

$$ax+b = a(x+2c)+b \quad\Rightarrow\quad a=0,$$

so all periodic, convex functions have the form $f(x) = b$.