STOCKHOLM UNIVERSITY
DEPARTMENT OF MATHEMATICS
Thomas Önskog

Written exam
Linear analysis
February 18, 2015

No calculator, book or notes are allowed. Complete justifications of your answers should be included. The problems are not arranged in order of difficulty. Results from a subproblem may be used in a subsequent subproblem even if the earlier subproblem has not been solved.

To pass the exam, the sum of the obtained C- and T-points should be no less than 15. Detailed information regarding the grading is found at the course web page. Optional oral exams will take place within two weeks from today for students that have scored at least 11 C-points and at least 11 T-points on this exam. Students that wish to take the oral exam should send an e-mail to luger@math.su.se no later than on February 25.

1. Consider the function (2C 3T)

$$f(x) = \begin{cases} \cos x, & \text{for } -\pi < x < 0, \\ 0, & \text{for } 0 < x < \pi. \end{cases}$$

   (a) Determine the Fourier series of f on $[-\pi, \pi]$.

   (b) Let g be the Fourier series of f on $[-\pi, \pi]$. Sketch the graph of g on the interval $[-2\pi, 2\pi]$. Be particularly clear at jumping points.

   (c) Let $h(x) = g(x) - \frac{1}{2}\cos x$. Determine if h is an odd or even function, determine the period of h and find an interval on which h converges uniformly to the function $f(x) - \frac{1}{2}\cos x$.

2. (a) Determine the cosine series of $f(x) = e^x$ on $[0, \pi]$. (3C 2T)

   (b) Calculate the sums

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2}.$$

3. Consider the Sturm-Liouville problem (3C 2T)

$$\begin{cases} xX''(x) + \frac{1}{2}X'(x) + \lambda X(x) = 0, & \text{for } 0 < x < 1, \\ X(0) = X'(1) = 0. \end{cases}$$

   (a) Use the change of variables $x = t^2$ to determine the eigenvalues and eigenfunctions of this Sturm-Liouville problem.

   (b) Formulate the orthonormality property of the eigenfunctions and check it directly.

4. (a) Let $g(x) = f(ax)$, for some constant $a > 0$. Prove the shift formula (2C 3T)

$$\mathcal{F}[g](\omega) = \frac{1}{a}\mathcal{F}[f]\left(\frac{\omega}{a}\right).$$

   (b) Use the fact $\mathcal{F}\left[e^{-|x|}\right](\omega) = \frac{1}{\pi(1+x^2)}$ to calculate the Fourier transform of $g(x) = \frac{1}{x^2+b^2}$.

   (c) Solve the integral equation

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + a^2}\,dt = \frac{1}{x^2+b^2}.$$

Please turn over!

5. Consider the function (3C 2T)

$$f(x) = \begin{cases} \frac{h}{a}x, & \text{for } 0 < x < a, \\ \frac{h}{1-a}(1-x), & \text{for } a < x < 1. \end{cases}$$

(a) Determine the sine series of f on $[0,1]$ and motivate why it converges uniformly to f on this interval.

(b) Use separation of variables to show that the boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, & \text{for } t > 0, \\ u(x,0) = f(x), \; u_t(x,0) = 0, & \text{for } 0 < x < 1, \end{cases}$$

is solved by

$$u(x,t) = \frac{2h}{\pi^2 a(1-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi a) \cos(n\pi ct) \sin(n\pi x).$$

6. (a) Let α and β be real positive constants. Prove that the function (2C 3T)

$$(f,g) = \int_{-1}^{1} f(x) g(x) (1-x)^{\alpha} (1+x)^{\beta} \, dx,$$

defines an inner product on the space of real, continuous functions on $[-1,1]$.

(b) Let $P_n^{(\alpha,\beta)}(x)$, $n \geq 0$, be a polynomial of degree n, such that $\left\{P_n^{(\alpha,\beta)}(x)\right\}$ form an orthonormal system on $[-1,1]$ with respect to the inner product in a. Find $P_0^{(1,1)}(x)$, $P_1^{(1,1)}(x)$ and $P_2^{(1,1)}(x)$.

The marked exams can be collected at the Student Affairs office, rooms 203–204, house 6. To get the exam result by e-mail, please send a message to luger@math.su.se.

Good luck!

Useful trigonometric formulae

$$\sin(\pi - \alpha) = \sin\alpha, \qquad \sin n\pi = 0, \text{ for } n \in \mathbb{Z}, \qquad \sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right),$$

$$\cos(\pi - \alpha) = -\cos\alpha, \qquad \cos n\pi = (-1)^n, \text{ for } n \in \mathbb{Z}, \qquad \tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right),$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}, \qquad \sin^2\alpha + \cos^2\alpha = 1,$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)), \\ \sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta)), \\ \cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)), \end{aligned}$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}, \qquad \sin^3\alpha = \frac{3\sin\alpha - \sin 3\alpha}{4}, \qquad \sin^4\alpha = \frac{3 - 4\cos 2\alpha + \cos 4\alpha}{8},$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}, \qquad \cos^3\alpha = \frac{3\cos\alpha + \cos 3\alpha}{4}, \qquad \cos^4\alpha = \frac{3 + 4\cos 2\alpha + \cos 4\alpha}{8},$$