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Written exam
Linear analysis
February 18, 2015

1. Consider the function (2C 3T)

$$f(x)=\begin{cases}\cos x, & \text{for } -\pi<x<0,\\ 0, & \text{for } 0<x<\pi.\end{cases}$$

(a) Determine the Fourier series of f on $[-\pi,\pi]$.
Solution: The Fourier series of f on $[-\pi,\pi]$ is given by

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right),$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \quad\text{and}\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

We first calculate a_0 as

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{0}\cos x\,dx=\frac{1}{\pi}\left[\sin x\right]_{-\pi}^{0}=0.$$

For a_n and b_n we have to treat the cases $n=1$ separately. For $n\neq1$ we get

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}\cos x\cos nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{0}\left(\cos(n+1)x+\cos(n-1)x\right)dx\\&=\frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_{-\pi}^{0}=0,\end{aligned}$$

and

$$\begin{aligned}b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}\cos x\sin nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{0}\left(\sin(n+1)x+\sin(n-1)x\right)dx\\&=-\frac{1}{2\pi}\left[\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_{-\pi}^{0}=-\frac{1}{2\pi}\left(\frac{1-(-1)^{n+1}}{n+1}+\frac{1-(-1)^{n-1}}{n-1}\right).\end{aligned}$$

For odd n, b_n vanishes, but for $n=2m$, we get

$$b_{2m}=-\frac{1}{\pi}\left(\frac{1}{2m+1}+\frac{1}{2m-1}\right)=-\frac{4m}{\pi\left(4m^2-1\right)}.$$

Finally, for $n=1$ we get

$$a_1=\frac{1}{\pi}\int_{-\pi}^{0}\cos^2 x\,dx=\frac{1}{\pi}\int_{-\pi}^{0}\frac{1+\cos 2x}{2}dx=\frac{1}{\pi}\left[\frac{x}{2}+\frac{\sin 2x}{4}\right]_{-\pi}^{0}=\frac{1}{2},$$

and

$$b_1=\frac{1}{\pi}\int_{-\pi}^{0}\cos x\sin x\,dx=\frac{1}{2\pi}\int_{-\pi}^{0}\sin 2x\,dx=-\frac{1}{2\pi}\left[\frac{\cos 2x}{2}\right]_{-\pi}^{0}=0.$$

Conclusively the Fourier series of f is

$$f(x) \sim \frac{1}{2}\cos x - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin(2mx).$$

(b) Let g be the Fourier series of f on $[-\pi,\pi]$. Sketch the graph of g on the interval $[-2\pi,2\pi]$. Be particularly clear at jumping points.
Solution: Since f is a continuously differentiable function, it satisfies the prerequisites of Dirichlet's theorem. Hence g converges pointwise to f at all points where f is continuous. At the points of discontinuity, that is $x=0$ and $x=\pm\pi$, g equals the mean value of its left and right limits which $1/2$ and $-1/2$ respectively. Moreover, g is 2π-periodic. Below, we sketch the graph of g on $[-2\pi,2\pi]$.

(c) Let $h(x) = g(x) - \frac{1}{2}\cos x$. Determine if h is an odd or even function, determine the period of h and find an interval on which h converges uniformly to the function $f(x) - \frac{1}{2}\cos x$.
Solution: On the interval $(-\pi,0)$, we have $f(x) - \frac{1}{2}\cos x = \frac{1}{2}\cos x$ and on the interval $(0,\pi)$, we have $f(x) - \frac{1}{2}\cos x = -\frac{1}{2}\cos x$. Since both f and $\frac{1}{2}\cos x$ are piecewise differentiable functions h will converge pointwise to this function, except for points of discontinuity, that is at $x=0$ and $x=\pm\pi$, and at these points the mean value of the left and right limits is zero. Hence h is a 2π-periodic function defined as

$$h(x) = \begin{cases} \frac{1}{2}\cos x, & \text{for } -\pi < x < 0, \\ -\frac{1}{2}\cos x, & \text{for } 0 < x < \pi, \\ 0, & \text{at } x=0 \text{ and } x=\pm\pi. \end{cases}$$

It follows that h is an odd function with period π and it will converge uniformly to $f(x) - \frac{1}{2}\cos x$ on all closed intervals not containing any of the points of discontinuity $x = n\pi$, where n is an integer.

2. (a) Determine the cosine series of $f(x) = e^x$ on $[0,\pi]$. (3C 2T)
Solution: The cosine series of f on $[0,\pi]$ is defined as

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

To determine a_n, we calculate

$$a_0 = \frac{2}{\pi}\int_0^{\pi} e^x dx = \frac{2}{\pi}\left[e^x\right]_0^{\pi} = \frac{2(e^{\pi}-1)}{\pi},$$

and, partially integrating twice,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^x \cos nx dx = \frac{2}{\pi}\left(\left[e^x \frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi e^x \frac{\sin nx}{n} dx\right) \\ &= -\frac{2}{n\pi}\left(\left[e^x \frac{-\cos nx}{n}\right]_0^\pi - \int_0^\pi e^x \frac{-\cos nx}{n} dx\right) = -\frac{2}{n\pi}\left(\frac{-e^\pi (-1)^n + 1}{n} + \frac{\pi}{2n} a_n\right), \end{aligned}$$

so that

$$a_n \left(1 + \frac{1}{n^2}\right) = \frac{2}{n^2\pi}\left(e^\pi (-1)^n - 1\right) \quad \Rightarrow \quad a_n = \frac{2\left(e^\pi (-1)^n - 1\right)}{\pi\left(n^2+1\right)}.$$

(b) Calculate the sums

$$\sum_{k=1}^\infty \frac{1}{1+k^2} \quad \text{and} \quad \sum_{k=1}^\infty \frac{(-1)^k}{1+k^2}.$$

Solution: The even extension of e^x from $[0,\pi]$ to $[-\pi,\pi]$ is piecewise differentiable and continuous and takes the same values at π and $-\pi$, so by Dirichlet's theorem, the cosine series converges pointwise on $[0,\pi]$. Let S_1 and S_2 denote the values of the two series above. Then, at $x=0$ and $x=\pi$, we have

$$1 = g(0) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n = \frac{e^\pi - 1}{\pi} + \sum_{n=1}^\infty \frac{2\left(e^\pi (-1)^n - 1\right)}{\pi\left(n^2+1\right)} = \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}\left(e^\pi I_2 - I_1\right),$$

and

$$e^\pi = g(\pi) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n (-1)^n = \frac{e^\pi - 1}{\pi} + \sum_{n=1}^\infty \frac{2\left(e^\pi - (-1)^n\right)}{\pi\left(n^2+1\right)} = \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}\left(e^\pi I_1 - I_2\right).$$

Multiplying the first relation with e^π and adding it to the second relation gives

$$2e^\pi = \frac{e^\pi - 1}{\pi} + e^\pi \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}\left(e^{2\pi} - 1\right) I_2 \quad \Rightarrow \quad I_2 = \frac{\pi e^\pi}{e^{2\pi} - 1} - \frac{1}{2} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right).$$

The first relation then gives

$$\begin{aligned} I_1 &= e^\pi I_2 + \frac{\pi}{2}\left(\frac{e^\pi - 1}{\pi} - 1\right) = e^\pi\left(\frac{\pi e^\pi}{e^{2\pi} - 1} - \frac{1}{2}\right) + \frac{e^\pi - 1}{2} - \frac{\pi}{2} = \\ &= \frac{2\pi e^{2\pi}}{2\left(e^{2\pi} - 1\right)} - \frac{1}{2} - \frac{\pi\left(e^{2\pi} - 1\right)}{2\left(e^{2\pi} - 1\right)} = \frac{\pi\left(e^{2\pi} + 1\right)}{2\left(e^{2\pi} - 1\right)} - \frac{1}{2} = \frac{1}{2}\left(\pi \tanh \pi - 1\right). \end{aligned}$$

3. Consider the Sturm-Liouville problem (3C 2T)

$$\begin{cases} xX''(x) + \frac{1}{2}X'(x) + \lambda X(x) = 0, & \text{for } 0 < x < 1, \\ X(0) = X'(1) = 0. \end{cases}$$

(a) Use the change of variables $x = t^2$ to determine the eigenvalues and eigenfunctions of this Sturm-Liouville problem.
Solution: Let $X(x) = X\left(t^2\right) := Y(t)$. Then

$$\begin{aligned} Y'(t) &= \frac{d}{dt} X\left(t^2\right) = X'(x)\, 2t, \\ Y''(t) &= \frac{d}{dt}\left(X'(x)\, 2t\right) = 2X'(x) + X''(x)\, 4t^2 = 4\left(xX''(x) + \frac{1}{2}X'(x)\right), \end{aligned}$$

so that the Sturm-Liouville problem, in terms of Y, can be written

$$\begin{cases} Y''(t) + \frac{\lambda}{4} Y(t) = 0, & \text{for } 0 < x < 1, \\ Y(0) = Y'(1) = 0. \end{cases}$$

By the lemma on non-negativity of eigenvalues of certain Sturm-Liouville problems, the eigenvalues are non-negative, so we investigate the two cases: $\lambda = 0$ and $\lambda/4 = \omega^2 > 0$. In the case of zero eigenvalues, the general solution is

$$Y(t) = at + b.$$

The boundary conditions give $Y(0) = b = 0$ and $Y'(1) = a = 0$, which corresponds to the trivial solution only. In the case of positive eigenvalues, the general solution is

$$Y(t) = a \cos \omega t + b \sin \omega t.$$

The first boundary condition gives $Y(0) = a = 0$ and hence $Y(t) = b \sin \omega t$. The second boundary condition now gives

$$Y'(1) = b\omega \cos \omega = 0,$$

so that a non-trivial solution must satisfy

$$\omega = \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi,$$

for some positive integer n. The eigenvalues to the Sturm-Liouville problem are given by $\lambda_n = 4\omega_n^2 = (2n+1)^2$ and the corresponding eigenfunctions are

$$Y_n(t) = A_n \sin \frac{2n+1}{2}\pi t \quad \Rightarrow \quad X_n(t) = A_n \sin \frac{2n+1}{2}\pi\sqrt{x}.$$

(b) Formulate the orthonormality property of the eigenfunctions and check it directly.
Solution: The orthonormality property is

$$(Y_m(t), Y_n(t)) = \int_0^1 Y_m(t) Y_n(t) \frac{1}{4} dt = \delta_{mn}.$$

For $m \neq n$, we have

$$\begin{aligned} (Y_m(t), Y_n(t)) &= \int_0^1 A_m \sin\left(\frac{2m+1}{2}\pi t\right) A_n \sin\left(\frac{2n+1}{2}\pi t\right) \frac{1}{4} dt \\ &= \frac{A_m A_n}{8} \int_0^1 \left(\cos\left((m-n)\pi t\right) + \cos\left((m+n+1)\pi t\right)\right) dt \\ &= \frac{A_m A_n}{8} \left[\frac{\sin(m-n)\pi t}{(m-n)\pi} + \frac{\sin(m+n+1)\pi t}{(m+n+1)\pi}\right]_0^1 = 0, \end{aligned}$$

and, for $m = n$, we have

$$\begin{aligned} (Y_m(t), Y_m(t)) &= \frac{A_m^2}{4} \int_0^1 \sin^2\left(\frac{2m+1}{2}\pi t\right) dt = \frac{A_m^2}{8} \int_0^1 (1 + \cos(2m+1)\pi t)\, dt \\ &= \frac{A_m^2}{8} \left[t + \frac{\sin(2m+1)\pi t}{(2m+1)\pi}\right]_0^1 = \frac{A_m^2}{8}, \end{aligned}$$

so that $A_m = 2\sqrt{2}$, and the normalized eigenfunctions are

$$X_n(t) = 2\sqrt{2} \sin \frac{2n+1}{2}\pi\sqrt{x}.$$

4. (a) Let $g(x) = f(ax)$, for some constant $a > 0$. Prove the shift formula (2C 3T)

$$\mathcal{F}[g](\omega) = \frac{1}{a}\mathcal{F}[f]\left(\frac{\omega}{a}\right).$$

Solution: The Fourier transform of g is defined as

$$\mathcal{F}[g](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\, e^{-i\omega x} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(ax)\, e^{-i\omega x} dx.$$

With the change of variables $y = ax$, $dy = adx$, we get

$$\mathcal{F}[g](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\, e^{-i\omega(y/a)} \frac{dy}{a} = \frac{1}{a}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\, e^{-i\left(\frac{\omega}{a}\right)y} dy\right) = \frac{1}{a}\mathcal{F}[f]\left(\frac{\omega}{a}\right).$$

(b) Use the fact $\mathcal{F}\left[e^{-|x|}\right](\omega) = \dfrac{1}{\pi(1+x^2)}$ to calculate the Fourier transform of $g(x) = \dfrac{1}{x^2+b^2}$.

Solution: The Fourier transform of $f(x) = e^{-|x|}$ is $\dfrac{1}{\pi(1+x^2)}$, so by the inverse Fourier transform, using the change of variables $y = -\omega, dy = -d\omega$,

$$e^{-|x|} = \text{P.V.}\int_{-\infty}^{\infty} \frac{1}{\pi(1+\omega^2)} e^{i\omega x} d\omega = \text{P.V.}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+y^2} e^{-iyx} dy,$$

which shows that the Fourier transform of $\dfrac{2}{1+x^2}$ is $e^{-|\omega|}$. Using the shift formula, it then follows that the Fourier transform of

$$g(x) = \frac{1}{x^2+b^2} = \frac{1}{b^2}\frac{1}{1+(x/b)^2} = \frac{1}{2b^2}\frac{2}{1+(x/b)^2},$$

is

$$G(\omega) = \frac{1}{2b^2}\frac{1}{1/b}e^{-\left|\frac{\omega}{1/b}\right|} = \frac{1}{2b}e^{-b|\omega|}.$$

(c) Solve the integral equation

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + a^2} dt = \frac{1}{x^2+b^2}.$$

Solution: The left hand side is the convolution between the unknown function $f(x)$ and the function $g(x) = \dfrac{1}{x^2+a^2}$. From the well-known result on Fourier transforms of convolutions, we get

$$\mathcal{F}[f](\omega)\,\mathcal{F}\left[\frac{1}{x^2+a^2}\right](\omega) = \mathcal{F}[f](\omega)\,\mathcal{F}[g](\omega) = \mathcal{F}[f*g](\omega) = \mathcal{F}\left[\frac{1}{x^2+b^2}\right](\omega),$$

and hence

$$\mathcal{F}[f](\omega) = \frac{\frac{1}{2b}e^{-b|\omega|}}{\frac{1}{2a}e^{-a|\omega|}} = \frac{2a(b-a)}{b}\frac{1}{2(b-a)}e^{-(b-a)|\omega|},$$

which implies that

$$f(x) = \frac{2a(b-a)}{b}\frac{1}{x^2+(b-a)^2}.$$

5. Consider the function (3C 2T)

$$f(x) = \begin{cases} \frac{h}{a}x, & \text{for } 0 < x < a, \\ \frac{h}{1-a}(1-x), & \text{for } a < x < 1. \end{cases}$$

(a) Determine the sine series of f on $[0,1]$ and motivate why it converges uniformly to f on this interval.

Solution: The sine series of f on $[0,1]$ is defined as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x, \quad \text{where} \quad b_n = 2\int_0^{\pi} f(x) \sin n\pi x dx.$$

We calculate the integral for b_n

$$\begin{aligned} b_n &= 2\int_0^{\pi} f(x)\sin n\pi x dx = 2\left(\int_0^a \frac{h}{a} x \sin n\pi x dx + \int_a^1 \frac{h}{1-a}(1-x)\sin n\pi x dx\right) \\ &= 2\left(\left[\frac{h}{a}x\frac{-\cos n\pi x}{n\pi}\right]_0^a - \int_0^a \frac{h}{a}\left(\frac{-\cos n\pi x}{n\pi}\right)dx\right) \\ &\quad +2\left(\left[\frac{h}{1-a}(1-x)\frac{-\cos n\pi x}{n\pi}\right]_a^1 - \int_a^1 \frac{h}{1-a}(-1)\left(\frac{-\cos n\pi x}{n\pi}\right)dx\right) \\ &= \frac{-2h\cos n\pi a}{n\pi} + \frac{2h}{a(n\pi)^2}[\sin n\pi x]_0^a + \frac{2h\cos n\pi a}{n\pi} - \frac{2h}{(1-a)(n\pi)^2}[\sin n\pi x]_a^1 \\ &= \left(\frac{2h}{a(n\pi)^2} + \frac{2h}{(1-a)(n\pi)^2}\right)\sin n\pi a = \frac{2h}{a(1-a)(n\pi)^2}\sin n\pi a. \end{aligned}$$

Since f is continuous, $f(0) = f(1)$ and $f' \in E$, the sine series converges uniformly to f on $[0,1]$.

(b) Use separation of variables to show that the boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, & \text{for } t > 0, \\ u(x,0) = f(x),\ u_t(x,0) = 0, & \text{for } 0 < x < 1, \end{cases}$$

has the formal solution

$$u(x,t) = \frac{2h}{\pi^2 a(1-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi a)\cos(n\pi ct)\sin(n\pi x).$$

Solution: Separating the variables as $v(x,t) = X(x)T(t)$, we get

$$X(x)T''(t) - c^2 X''(x)T(t) = 0 \quad \Rightarrow \quad \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

which corresponds to the Sturm-Liouville problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, & \text{for } 0 < x < 1, \\ X(0) = X(1) = 0. \end{cases}$$

and the differential equation $T''(t) + \lambda c^2 T(t) = 0$. By the lemma on non-negativity of solutions to certain Sturm-Liouville problems, we know that the possibilities are either $\lambda = \omega^2 > 0$ or $\lambda = 0$.

The case $\lambda = 0$ corresponds to $X(x) = ax + b$, but an application of the boundary conditions immediately gives that only the trivial solution can exist in this case. In the case $\lambda > 0$, the general solution is

$$X(x) = a\cos\omega x + b\sin\omega x.$$

The first boundary condition gives $a = 0$ and the second boundary condition then gives $b\sin\omega = 0$, which has nontrivial solutions for $\omega = n\pi$, for some positive integer n. The eigenvalues to the Sturm-Liouville problem are given by $\lambda_n = (n\pi)^2$ and the corresponding eigenfunctions by $X_n(x) = \sin n\pi x$. The corresponding functions $T_n(t)$ satisfy the differential equation $T_n''(t) + (n\pi c)^2 T_n(t) = 0$ and can thus be written

$$T_n(t) = d_n\cos(n\pi ct) + e_n\sin(n\pi ct).$$

We now assume that $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$. The first initial condition then gives

$$u(x,0) = \sum_{n=1}^{\infty} d_n \sin n\pi x = f(x) = \sum_{n=1}^{\infty} \frac{2h}{a(1-a)(n\pi)^2}\sin n\pi a \sin n\pi x,$$

so that

$$d_n = \frac{2h}{a(1-a)(n\pi)^2}\sin n\pi a.$$

Similarly, the second initial condition gives

$$u_t(x,0) = \sum_{n=1}^{\infty} e_n n\pi c \sin n\pi x = 0 \quad\Rightarrow\quad e_n = 0.$$

Conclusively, the formal solution is, as expected,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty}\left(\frac{2h}{a(1-a)(n\pi)^2}\sin n\pi a\cos(n\pi ct)\right)(\sin n\pi x).$$

6. (a) Let α and β be real positive constants. Prove that the function (2C 3T)

$$(f,g) = \int_{-1}^{1} f(x)\,g(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx,$$

defines an inner product on the space of real, continuous functions on $[-1,1]$.

Solution: We need to show that $(f,f) \geq 0$, with equality if and only if $f = 0$, that $(f,g) = (g,f)$ and that $(af + bg, h) = a(f,h) + b(g,h)$. We first investigate (f,f), which equals

$$(f,f) = \int_{-1}^{1} (f(x))^2 (1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Since $(f(x))^2$, $1-x$ and $1+x$ are all non-negative on $(-1,1)$, the integrand is non-negative and so is the integral. Since the integrand is a non-negative, continuous function, the integral can only be zero if the integrand is identically zero and this implies that $f = 0$. Indeed, if $f \neq 0$ holds at some point, by continuity, it must hold at an interval surrounding that point and, if so, the integral cannot be zero. Next, we show

$$(f,g) = \int_{-1}^{1} f(x)\,g(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx = \int_{-1}^{1} g(x)\,f(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx = (g,f),$$

and finally, by the linearity of the integral

$$
\begin{aligned}
(af+bg,h) &= \int_{-1}^{1}(af(x)+bg(x))\,h(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx\\
&= \int_{-1}^{1} af(x)\,h(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx+\int_{-1}^{1} bg(x)\,h(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx\\
&= a(f,h)+b(g,h).
\end{aligned}
$$

(b) Let $P_n^{(\alpha,\beta)}(x)$, $n\ge 0$, be a polynomial of degree n, such that $\left\{P_n^{(\alpha,\beta)}(x)\right\}$ form an orthonormal system on $[-1,1]$ with respect to the inner product in a. Find $P_0^{(1,1)}(x)$, $P_1^{(1,1)}(x)$ and $P_2^{(1,1)}(x)$.
Solution: We use the Gram-Schmidt process starting with the basis $\{1,x,x^2\}$ to construct an orthonormal system with respect to the inner product

$$(f,g)=\int_{-1}^{1} f(x)\,g(x)\,(1-x)^{1}(1+x)^{1}\,dx=\int_{-1}^{1} f(x)\,g(x)\left(1-x^2\right)dx.$$

We normalize the first basis function

$$\|1\|^2=\int_{-1}^{1}1^2\left(1-x^2\right)dx=\left[x-\frac{x^3}{3}\right]_{-1}^{1}=\frac{4}{3},$$

so the first normalized basis function is $P_0^{(1,1)}(x)=\sqrt{3}/2$. We next find the orthogonal projection of x on $P_0^{(1,1)}(x)$, that is

$$\left(x,P_0^{(1,1)}(x)\right)P_0^{(1,1)}(x)=\left(\int_{-1}^{1}x\frac{\sqrt{3}}{2}\left(1-x^2\right)dx\right)\frac{\sqrt{3}}{2}=\frac{3}{4}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{-1}^{1}=0,$$

so x and $P_0^{(1,1)}(x)$ are already orthogonal. We normalize the second basis function

$$\|x\|^2=\int_{-1}^{1}x^2\left(1-x^2\right)dx=\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_{-1}^{1}=\frac{4}{15},$$

so the second normalized basis function is $P_1^{(1,1)}(x)=\sqrt{15}x/2$. We next find the orthogonal projection of x^2 on the space spanned by $P_0^{(1,1)}(x)$ and $P_1^{(1,1)}(x)$, that is

$$
\begin{aligned}
&\left(x^2,P_0^{(1,1)}(x)\right)P_0^{(1,1)}(x)+\left(x^2,P_1^{(1,1)}(x)\right)P_1^{(1,1)}(x)\\
&=\left(\int_{-1}^{1}x^2\frac{\sqrt{3}}{2}\left(1-x^2\right)dx\right)\frac{\sqrt{3}}{2}+\left(\int_{-1}^{1}x^2\frac{\sqrt{15}}{2}x\left(1-x^2\right)dx\right)\frac{\sqrt{15}}{2}x\\
&=\frac{3}{4}\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_{-1}^{1}+\frac{15}{4}x\left[\frac{x^4}{4}-\frac{x^6}{6}\right]_{-1}^{1}=\frac{3}{4}\frac{4}{15}=\frac{1}{5},
\end{aligned}
$$

so the third basis function is proportional to $x^2-1/5$. Normalizing, we get

$$
\begin{aligned}
\left\|5x^2-1\right\|^2 &= \int_{-1}^{1}\left(5x^2-1\right)^2\left(1-x^2\right)dx=\int_{-1}^{1}\left(25x^4-10x^2+1\right)\left(1-x^2\right)dx\\
&= \int_{-1}^{1}\left(-25x^6+35x^4-11x^2+1\right)=\left[-\frac{25x^7}{7}+\frac{35x^5}{5}-\frac{11x^3}{3}+x\right]_{-1}^{1}=\frac{32}{21},
\end{aligned}
$$

so the third normalized eigenfuntion is $P_2^{(1,1)}(x)=\sqrt{21/32}\left(5x^2-1\right)$.