Matematiska Institionen, Avd. Matematik, Stockholms universitet, Tentamenskrivning i Linjär analys, 2016-01-12

Examinator : Rostom Getsadze

1. (a). Find the Fourier series of the function

$$f(x) = \mid \sin x \mid, \ \ -\pi < x \leq \pi.$$

(b) Calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

(c) Calculate the sum

$$1 + \sum_{n=1}^{\infty} \frac{4}{(4n^2 - 1)^2}.$$

5p

2. On the linear space $C([0, 2\pi])$ we define the inner product

$$< f, g >= \int_0^{2\pi} f(x)\overline{g(x)}dx.$$

(a) Prove that the set

$$S = \{\frac{1}{\sqrt{2\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}\}$$

is an orthonormal system on $[0, 2\pi]$.

(b) Let W be the subspace spanned by S, and let $f(x) = x$ on the interval $[0, 2\pi]$. Find the function g in W which is closest to f (that is to say, for which $\parallel f - g \parallel$ is minimal).

5p

3. Let E denote the space of all piecewise continuous, complex-valued functions on $[-\pi, \pi]$.

Let $E^{'}$ denote the space of all complex-valued functions f which satisfy the following conditions:

1. $f \in E$.
2. At each $x \in [-\pi, \pi)$, the following limit exists (and is finite):

$$\lim_{h \to 0+} \frac{f(x+h) - f(x+)}{h}$$

3. At each $x \in (-\pi, \pi]$, the following limit exists (and is finite):

$$\lim_{h \to 0+} \frac{f(x-) - f(x-h)}{h}$$

Prove that if $f^{'} \in E$ then $f \in E^{'}$.

5p

4. Let L be the Sturm-Liouville differential operator defined on $C^2([a, b])$ by

$$(Lu)(x) = -(p(x)u^{'}(x))^{'} + q(x)u(x),$$

where $p(x)$ and $q(x)$ are real-valued functions.
Let $u, v \in C^2([a, b])$ satisfy the following conditions

$$\alpha_1 u(a) + \alpha_2 u^{'}(a) = 0$$

and

$$\beta_1 u(b) + \beta_2 u^{'}(b) = 0,$$

for some real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $(\alpha_1, \alpha_2) \neq (0, 0)$ and $(\beta_1, \beta_2) \neq (0, 0)$.
Prove that then

$$< Lu, v > = < u, Lv > .$$

5p

5. (a) Find the Fourier transform of the function

$$g(t) = e^{-|t|}, \qquad t \in (-\infty, +\infty).$$

(b) What function f has the Fourier transform

$$\hat{f}(w) = \frac{1}{(1+w^2)^2}, \qquad w \in (-\infty, +\infty)?$$

5p

6. Find the eigenvalues and eigenfunctions to the following Sturm-Liouville problem

$$x^2 X''(x) + xX'(x) + \lambda X(x) = 0, \quad x \in [1, e^2],$$

subject to the boundary condition

$$X(1) = X(e^2) = 0.$$

Hint: use the following substitution

$$x = e^s.$$

5p

Trigonometric Formulae

$$\sin(\pi - \alpha) = \sin \alpha$$

$$\sin n\pi = 0, \;\; n \in \mathbf{Z}$$

$$\cos(\pi - \alpha) = -\cos \alpha$$

$$\cos n\pi = (-1)^n, \;\; n \in \mathbf{Z}$$

$$\sin \alpha = \cos(\frac{\pi}{2} - \alpha)$$

$$\tan \alpha = \cot(\frac{\pi}{2} - \alpha)$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$$

Vänd

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$
$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$
$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$$
$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$$