Matematiska Institutionen, Avd. Matematik, Stockholms universitet, Tentamenskrivning i Linjär analys, 2016-01-12, Lösningar

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1. (a). Find the Fourier series of the function

$$f(x) = \mid \sin x \mid, \quad -\pi < x \leq \pi.$$

(b) Calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

(c) Calculate the sum

$$1 + \sum_{n=1}^{\infty} \frac{4}{(4n^2 - 1)^2}.$$

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Solution. Let

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

It is clear that since $f(x) \sin nx$ is even and $f(x) \cos nx$ is odd on $[-\pi, \pi]$ we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mid \sin t \mid \sin ntdt = 0$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mid \sin t \mid \cos ntdt = \frac{2}{\pi} \int_0^{\pi} \sin t \cos ntdt$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(n+1)tdt - \frac{1}{\pi} \int_0^{\pi} \sin(n-1)tdt =$$

$$= -\frac{1}{\pi}\frac{\cos(n+1)t}{n+1}\,|_0^\pi + \frac{1}{\pi}\frac{\cos(n-1)t}{n-1}\,|_0^\pi$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{4}{\pi(1-4k^2)}, & \text{if } n = 2k \text{ is even.} \end{cases}.$$

Thus

$$f(x) \sim \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}.$$

When $x = 0$ we get from Dirichlet's theorem (f is continuous at 0)

$$0 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{1-4k^2}.$$

and, consequently,

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

According to Parceval's identity we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(t)dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

Consequently,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 t dt = \frac{1}{2}\frac{16}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}.$$

On the other hand

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 t dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2t)dt = 1$$

Thus

$$\frac{1}{2}\frac{16}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = 1$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}$$

Finally, we obtain

$$1 + \sum_{n=1}^{\infty}\frac{4}{(4n^2-1)^2} = \frac{\pi^2}{4} - 1.$$

2. On the linear space $C([0, 2\pi])$ we define the inner product

$$< f, g >= \int_0^{2\pi} f(x)\overline{g(x)}dx.$$

(a) Prove that the set

$$S = \{\frac{1}{\sqrt{2\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}\}$$

is an orthonormal system on $[0, 2\pi]$.

(b) Let W be the subspace spanned by S, and let $f(x) = x$ on the interval $[0, 2\pi]$. Find the function g in W which is closest to f (that is to say, for which $\| f - g \|$ is minimal).

Solution.

$$< \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} >= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}dx = 1.$$

$$< \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}} > \frac{1}{\pi}\int_0^{2\pi} \cos^2 2xdx = \frac{1}{2\pi}\int_0^{2\pi}(1 + \cos 4x)dx$$

$$= 1 + \frac{1}{2\pi}\frac{\sin 4x}{4} |_0^{2\pi}= 1.$$

$$< \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}} > \frac{1}{\pi}\int_0^{2\pi} \sin^2 2xdx = \frac{1}{2\pi}\int_0^{2\pi}(1 - \cos 4x)dx$$

$$= 1 - \frac{1}{2\pi}\frac{\sin 4x}{4} |_0^{2\pi}= 1.$$

$$< \frac{1}{\sqrt{2\pi}}, \frac{\cos 2x}{\sqrt{\pi}} >= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \cos 2xdx = \frac{1}{\sqrt{2\pi}}\frac{\sin 2x}{2} |_0^{2\pi}= 0.$$

$$< \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}} >= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin dx = -\frac{1}{\sqrt{2\pi}}\cos x |_0^{2\pi}= -\frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} = 0.$$

$$< \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}} >= \frac{1}{\pi}\int_0^{2\pi} \sin x \cos 2xdx = \frac{1}{2\pi}\int_0^{2\pi}(\sin(x - 2x) + \sin(x + 2x))dx =$$

$$= \frac{1}{2\pi}\cos x \mid_0^{2\pi} - \frac{1}{2\pi}\frac{\cos 3x}{3} \mid_0^{2\pi} = 0.$$

Consequently, S is an orthonormal system.

The function which is closest to $f(x) = x$ on the interval $[0, 2\pi]$ is the projection of f on W, that is, the function

$$\tilde{f}(x) = < \frac{1}{\sqrt{2\pi}}, f > \frac{1}{\sqrt{2\pi}} + < \frac{\cos 2x}{\sqrt{\pi}}, f > \frac{\cos 2x}{\sqrt{\pi}} + < \frac{\sin x}{\sqrt{\pi}}, f > \frac{\sin x}{\sqrt{\pi}}.$$

Now we have,

$$< \frac{1}{\sqrt{2\pi}}, f > = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} x dx = \frac{1}{\sqrt{2\pi}}\frac{1}{2}x^2 \mid_0^{2\pi} = \pi\sqrt{2\pi}.$$

$$< \frac{\cos 2x}{\sqrt{\pi}}, f > = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} x\cos 2x dx =$$

$$= \frac{1}{\sqrt{\pi}}\frac{x\sin 2x}{2} \mid_0^{2\pi} - \frac{1}{2\sqrt{\pi}}\int_0^{2\pi}\sin 2x dx = 0.$$

And, finally,

$$< \frac{\sin x}{\sqrt{\pi}}, f > = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} x\sin x dx =$$

$$= -\frac{1}{\sqrt{\pi}}x\cos x \mid_0^{2\pi} + \frac{1}{\sqrt{\pi}}\int_0^{2\pi}\cos x dx = -2\sqrt{\pi}.$$

Thus the function The function which is closest to $f(x) = x$ on the interval $[0, 2\pi]$ is the projection of f on W, that is, the function

$$\tilde{f}(x) = \pi\sqrt{2\pi}.\frac{1}{\sqrt{2\pi}} + 0\cdot\frac{\cos 2x}{\sqrt{\pi}} - 2\sqrt{\pi}\frac{\sin x}{\sqrt{\pi}} = \pi - 2\sin x.$$

3. Let E denote the space of all piecewise continuous, complex-valued functions on $[-\pi, \pi]$.

Let E' denote the space of all complex-valued functions f which satisfy the following conditions:

1. $f \in E$.

2. At each $x \in [-\pi, \pi)$, the following limit exists (and is finite):

$$\lim_{h \to 0+} \frac{f(x+h) - f(x+)}{h}$$

3. At each $x \in (-\pi, \pi]$, the following limit exists (and is finite):

$$\lim_{h \to 0+} \frac{f(x-) - f(x-h)}{h}$$

Prove that if $f^{'} \in E$ then $f \in E^{'}$.

Solution. Let $f^{'} \in E$. Then, according to the definition of E there exists a finite sequence of real numbers

$$-\pi = \delta_0 < \delta_1 < \ldots < \delta_k < \delta_{k+1} < \ldots < \delta_n = \pi,$$

such that $f^{'}$ is continuous on each interval (δ_k, δ_{k+1}) and $f^{'}$ has all posiible right and left limits at all δ_k.

We have to prove that $f \in E^{'}$.

At first we show that at each $-\pi \leq \delta_k < \pi$ the $\lim_{x \to \delta_k} f(x)$ exists. Indeed, Let

$$\delta_k < x_1 < x_2 < \delta < \delta_{k+1}.$$

Then according to the mean value theorem for derivatives, applied to the interval $[x_1, x_2]$, there exist a number $x_1 < \xi < x_2$ such that:

$$f(x_2) - f(x_1) = f^{'}(\xi)(x_2 - x_1).$$

It is clear that as $\delta \to \delta_k+$ then $\xi \to \delta_k+$ and, consequently, using the fact that $f^{'} \in E$ we obtain

$$\lim_{\delta \to \delta_k+} (f(x_2) - f(x_1)) = f^{'}(\delta_k+) \cdot 0.$$

Now using the Cauchy criteria we conclude that the $\lim_{x \to \delta_k} f(x)$ exists. Existence of right limits are proved similarly.

Next we show that at each $\delta_k \in [-\pi, \pi)$, the following limit exists (and is finite):

$$\lim_{h \to 0+} \frac{f(\delta_k + h) - f(\delta_k+)}{h}$$

Indeed, let $0 < h < \delta_{k+1} - \delta_k$ and $\delta_k < y < \delta_k + h$. Then, according to the Fundamental theorem of calculus

$$f(\delta_k + h) - f(y) = \int_y^{\delta_k + h} f^{'}(t)dt.$$

Now we take limit in this equality as $y \to \delta_k+$ and taking account of the fact that $f^{'}(t)$ is a bounded function on $[-\pi, \pi]$ we obtain

$$f(\delta_k + h) - f(\delta_k+) = \int_{\delta_k}^{\delta_k + h} f^{'}(t)dt.$$

But according to the mean value theorem for integrals, there exists a number η in the interval $[\delta_k, \delta_k + h]$ such that

$$\int_{\delta_k}^{\delta_k + h} f^{'}(t)dt = f^{'}(\eta)h.$$

From the last two formulas we can conclude, that

$$\frac{f(\delta_k + h) - f(\delta_k+)}{h} = f^{'}(\eta).$$

But when $h \to 0$ then $\eta \to \delta_k+$ and $f^{'}(\eta) \to f^{'}(\delta_k+)$. Consequently, the limit

$$\lim_{h \to 0+} \frac{f(\delta_k + h) - f(\delta_k+)}{h}$$

exists and is finite. The case of the left-sided derivatives can be proved similarly.

4. Let L be the Sturm-Liouville differential operator defined on $C^2([a, b])$ by

$$(Lu)(x) = -(p(x)u^{'}(x))^{'} + q(x)u(x),$$

where $p(x)$ and $q(x)$ are real-valued functions.

Let $u, v \in C^2([a, b])$ satisfy the following conditions

$$\alpha_1 u(a) + \alpha_2 u^{'}(a) = 0$$

and

$$\beta_1 u(b) + \beta_2 u^{'}(b) = 0,$$

for some real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $(\alpha_1, \alpha_2) \neq (0,0)$ and $(\beta_1, \beta_2) \neq (0,0)$. Prove that then

$$< Lu, v >=< u, Lv > .$$

Solution. Using the fact that p and q are real-valued, also partial integration and the boundary conditions we obtain

$$< Lu, v > - < u, Lv >= \int_a^b (-(pu^{'})^{'} + qu)\overline{v}dx - \int_a^b u\overline{(-(pv^{'})^{'} + qv)}dx$$

$$= \int_a^b (-(pu^{'})^{'}\overline{v} + qu\overline{v} + u\overline{(pv^{'})^{'}} - u\overline{qv})dx$$

$$= \int_a^b (-(pu^{'})^{'}\overline{v} + u(p(\overline{v})^{'})^{'}dx$$

$$= \left[-pu^{'}\overline{v} + up\overline{v}^{'}\right]_a^b + \int_a^b (pu^{'}(\overline{v})^{'} - u^{'}p(\overline{v})^{'})dx$$

$$= \left[-pu^{'}\overline{v} + up\overline{v}^{'}\right]_a^b + 0$$

Now we look at the boundary conditions $R_1 u = R_1 v = 0$, that is

$$\alpha_1 u(a) + \alpha_2 u^{'}(a) = 0$$

$$\alpha_1 v(a) + \alpha_2 v^{'}(a) = 0$$

The second equality implies

$$\alpha_1 \overline{v(a)} + \alpha_2 \overline{v'(a)} = 0$$

because α_1 and α_2 are real. According to the assumption, this homogeneous system of linear equations

$$\alpha_1 \overline{v(a)} + \alpha_2 \overline{v'(a)} = 0$$
$$\alpha_1 u(a) + \alpha_2 u^{'}(a) = 0$$

has non-trivial solution (α_1, α_2) and consequently, it's determinant is zero. Thus

$$u(a)\overline{v}^{'}(a) - \overline{v(a)}u^{'}(a) = 0.$$

Similarly, we can prove that

$$u(b)\overline{v}^{'}(b) - \overline{v(b)}u^{'}(b) = 0.$$

Consequently

$$\left[-pu'\overline{v} + up\overline{v}'\right]_a^b = 0.$$

Now it is clear from above formulas, that

$$< Lu, v > - < u, Lv >= 0.$$

The statement is now proven.

5. (a) Find the Fourier transform of the function

$$g(t) = e^{-|t|}, \qquad t \in (-\infty, +\infty).$$

(b) What function f has the Fourier transform

$$f(w) = \frac{1}{(1+w^2)^2}, \qquad w \in (-\infty, +\infty)?$$

Solution. (a) If

$$g(t) = e^{-|t|}, \qquad t \in (-\infty, +\infty)$$

then

$$\hat{g}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|t|}e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{0} e^{t}e^{-i\omega t}dt +$$
$$+\frac{1}{2\pi}\int_{0}^{\infty} e^{-t}e^{-i\omega t}dt = \frac{1}{2\pi}\frac{1}{1-i\omega} + \frac{1}{2\pi}\frac{1}{1+i\omega}$$

We note that for every real R

$$\mid e^{R(1-i\omega)} \mid = e^R \mid e^{-iR\omega} \mid = e^R$$

and, consequently,

$$\lim_{R\to-\infty} e^{R(1-i\omega)} = 0.$$

Similarly, we can show that

$$\lim_{R\to\infty} e^{-R(1+i\omega)} = 0.$$

These relations imply that

$$\hat{g}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{t(1-i\omega)}dt + \frac{1}{2\pi}\int_{0}^{\infty} e^{-t(1+i\omega)}dt$$

$$= \frac{1}{2\pi(1-i\omega)} e^{t(1-i\omega)} \mid_{-\infty}^{0} - \frac{1}{2\pi(1+i\omega)} e^{t(1+i\omega)} \mid_{0}^{\infty}$$

$$= \frac{1}{2\pi(1-i\omega)} + \frac{1}{2\pi(1+i\omega)} = \frac{1}{2\pi(1+\omega^2)}$$

On the other hand, according to the convolution theorem for Fourier transforms

$$F(g*g)(\omega) = 2\pi F(g)(\omega)F(g)(\omega) = 2\pi \frac{1}{2\pi(1+\omega^2)} \frac{1}{2\pi(1+\omega^2)} = \frac{1}{2\pi(1+\omega^2)^2}$$

Consequently, the function whose Fourier transform equals to $\frac{1}{(1+\omega^2)^2}$ is the function

$$h(x) = \frac{1}{2\pi}(g*g)(x).$$

Let us evaluate the $(g*g)(x)$. Let $x \geq 0$.

$$(g*g)(x) = \int_{-\infty}^{\infty} g(x-y)g(y)dy = \int_{-\infty}^{\infty} g(x-y)g(y)dy = \int_{-\infty}^{\infty} e^{-|x-y|}e^{-|y|}dy$$

$$= \int_{-\infty}^{0} + \int_{0}^{x} + \int_{x}^{\infty} =$$

$$= \int_{-\infty}^{0} e^{-(x-y)}e^{y}dy + \int_{0}^{x} e^{-(x-y)}e^{-y}dy + \int_{x}^{\infty} e^{x-y}e^{-y}dy =$$

$$= \int_{-\infty}^{0} e^{-x+2y}dy + \int_{0}^{x} e^{-x}dy + \int_{x}^{\infty} e^{x-2y}dy =$$

$$= e^{-x}\frac{1}{2}e^{2y} \mid_{-\infty}^{0} + e^{-x}y \mid_{0}^{x} - e^{x}\frac{1}{2}e^{-2y} \mid_{x}^{\infty} =$$

$$= e^{-x}\frac{1}{2} + e^{-x}x + e^{-x}\frac{1}{2} = e^{-x}(1+x).$$

Similarly we can get for $x < 0$

$$(g*g)(x) = e^{x}(1-x).$$

Consequently, the answer to b.) is the following function

$$2\pi(g*g)(x) = 2\pi e^{-|x|}(1+ \mid x \mid).$$

6. Find the eigenvalues and eigenfunctions to the following Sturm-Liouville problem

$$x^2 X''(x) + xX'(x) + \lambda X(x) = 0, \quad x \in [1, e^2],$$

subject to the boundary condition

$$X(1) = X(e^2) = 0.$$

Hint: use the following substitution

$$x = e^s.$$

Solution. Using the substitution $x = e^s$ we reduce the given problem to the following

$$Y''(s) + \lambda Y(s) = 0, \quad x \in [1, e^2],$$

$$Y(0) = Y(2) = 0.$$

This problem has spectrum

$$\alpha_n = \frac{\pi n}{2}$$

and eigenvectors

$$Y_n(s) = \sin \frac{\pi n s}{2}.$$

With respect to the original variable x we obtain

$$X_n(x) = \sin(\frac{\pi n \ln x}{2}).$$

Trigonometric Formulae

$$\sin(\pi - \alpha) = \sin \alpha$$

$$\sin n\pi = 0, \ \ n \in \mathbf{Z}$$

$$\cos(\pi - \alpha) = -\cos \alpha$$

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$$\cos n\pi = (-1)^n, \;\; n \in \mathbf{Z}$$
$$\sin\alpha = \cos(\frac{\pi}{2} - \alpha)$$
$$\tan\alpha = \cot(\frac{\pi}{2} - \alpha)$$
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$
$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$
$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$
$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$$
$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$$