DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
Section mathematics
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Written examination
Linear analysis
January 5, 2017

No aids are allowed. Answer in English or Swedish.

1. Consider the 2π-periodic function $f(x)$ on $\mathbf{R}$ for which $f(x) = x$, $0 < x \leq \pi$, and $f(x) = \pi - x$, $\pi < x \leq 2\pi$. Determine the Fourier series expansion of $f(x)$. Determine the sum of each of the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$
Determine the complex numbers a, b and c that minimize the value of the integral
$$\int_0^{2\pi} |f(x) - a\sin x - b\sin 2x - c\sin 3x|^2 \, dx$$
and determine the minimum value of the integral. 10 p

2. Determine all continuous functions $f(x)$ on $\mathbf{R}$ with $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ such that $f(x) + \int_{-\infty}^{\infty} f(x-y)\, e^{-|y|}\, dy = e^{-|x|}$ for all $x \in \mathbf{R}$. 10 p

3. a) Find the solution $u(x,t)$ to the heat equation $u_t - u_{xx} = 0$, $0 < x < 1$, $t > 0$, with initial value $u(x,0) = x$, $0 < x < 1$, and boundary values $u_x(0,t) = 0$ and $u_x(1,t) = 0$, $t > 0$.

   b) Find the solution $v(x,t)$ to the heat equation $v_t - v_{xx} = x$, $0 < x < 1$, $t > 0$, with initial value $v(x,0) = x$, $0 < x < 1$, and boundary values $v_x(0,t) = 0$ and $v_x(1,t) = 0$, $t > 0$. 10 p

4. Let $f(x)$ be an arbitrary complex-valued C^2-function on $0 \leq x \leq \pi$ such that $f(0) = 0$ and $f(\pi) = 0$. Let $f(x) \sim \sum_{n=1}^{\infty} a_n \sin nx$ be the sine series expansion of $f(x)$ on $0 \leq x \leq \pi$. Show that $f'(x) \sim \sum_{n=1}^{\infty} n a_n \cos nx$ is the cosine series expansion of the first derivative $f'(x)$ on $0 \leq x \leq \pi$, and that $f''(x) \sim -\sum_{n=1}^{\infty} n^2 a_n \sin nx$ is the sine series expansion of the second derivative $f''(x)$ on $0 \leq x \leq \pi$. Prove the inequalities
$$\int_0^{\pi} |f(x)|^2\, dx \leq \int_0^{\pi} |f'(x)|^2\, dx \quad \text{and} \quad \int_0^{\pi} |f'(x)|^2\, dx \leq \left(\int_0^{\pi} |f(x)|^2\, dx\right)^{1/2} \left(\int_0^{\pi} |f''(x)|^2\, dx\right)^{1/2}$$
and for each inequality determine all functions $f(x)$ giving equality in the inequality. 10 p

5. For $u \in \mathbf{R}$ show that
$$\frac{1}{2}\int_0^2 \sin ux\, dx - \frac{1}{8}\int_0^4 \sin ux\, dx = \begin{cases} \dfrac{\sin^4 u}{u} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0. \end{cases}$$
Show that the generalized integral
$$\int_0^{\infty} \frac{\sin^4 u}{u} \sin ux\, du$$
is convergent for each $x \in \mathbf{R}$ and determine the value of the generalized integral for each $x \in \mathbf{R}$. Determine the value of
$$\int_0^{\infty} \frac{\sin^8 u}{u^2}\, du.$$
10 p

Continuation on the other side.

6. Let $f(x)$ be an arbitrary complex-valued piecewise continuous function on $0 \le x \le 2\pi$ and let $g(x)$ be an arbitrary complex-valued 2π-periodic piecewise continuous function on $\mathbf{R}$. Let $g(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier series expansion of $g(x)$. Set $S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ for integers $N \ge 1$. Show that

$$\left| \int_0^{2\pi} f(x)\big(g(\lambda x) - S_N(\lambda x)\big)\,dx \right| \le \sqrt{2\pi} \left(\int_0^{2\pi} |f(x)|^2\,dx \right)^{1/2} \left(\sum_{n=N+1}^{\infty} \big(|a_n|^2 + |b_n|^2\big) \right)^{1/2}$$

for all real numbers $\lambda > 1$ and all integers $N \ge 1$. Show that

$$\int_0^{2\pi} f(x) g(\lambda x)\,dx \to \frac{1}{2\pi} \left(\int_0^{2\pi} f(x)\,dx \right) \left(\int_0^{2\pi} g(x)\,dx \right) \quad \text{as } \lambda \to \infty.$$

10 p

Return of examination scripts Monday January 16 between 11.45 and 12.00 in room 32 house 5, and thereafter in room 204 house 6.